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Free and based path groupoids

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We give an explicit description of the free path and loop groupoids in the Morita bicategory of translation topological groupoids. We prove that the free path groupoid of a discrete group acting properly on a topological space X is a translation groupoid given by the same group acting on the topological path space X^I . We give a detailed description of based path and loop groupoids and show that both are equivalent to topological spaces. We also establish the notion of homotopy and fibration in this context.

18B40, 55P35, 58E40; 55R91, 58D19

1 Introduction

Our aim is to give an explicit description of the path object in the bicategory of translation topological groupoids. Our main application will be in the setting of orbifolds as groupoids.

We adopt the model developed by Moerdijk and Pronk [9] to describe orbifolds in terms of groupoids. Essentially an orbifold is a Morita equivalence class of groupoids of a certain type, which we will call orbifold groupoids.

In this spirit, the right notion of morphism between orbifold groupoids is that of a generalized map. These generalized maps arise as morphisms in the bicategory of topological groupoids, functors and natural transformations when inverting the essential equivalences; see Pronk [13].

All orbifolds can be represented by a groupoid given by a certain type of action of a group G on a topological space X . This representation $G \ltimes X$ is called translation

groupoid. In particular we will be interested in *developable* orbifolds defined by a translation groupoid given by a discrete group acting properly on a space.

For these orbifolds, we use their groupoid characterization to obtain a description of the generalized maps from the interval to the orbifold as a translation groupoid. We prove that the free path groupoid of the translation groupoid $G \ltimes X$ is the translation groupoid $G \ltimes X^I$. In fact we describe three different approaches resulting in three characterizations of the path groupoid: as a colimit of G -paths, as a groupoid of multiple G -paths and as a translation groupoid $G \ltimes X^I$. We prove that the three groupoids are equivalent.

We show that this construction of the path groupoid is functorial and invariant under Morita equivalence.

The pullback along the diagonal of this model gives us as a particular case, the free loop groupoid which coincides with the descriptions given by Lupercio and Uribe [7], Adem, Leida and Ruan [1] and Noohi [10] in various contexts.

Moreover, we use this model to calculate the based groupoid of paths between two points. We prove that this groupoid is actually equivalent to a topological space.

Using our description of the path groupoid, we provide an explicit characterization for a homotopy between two generalized maps, as well as a definition of orbifold fibrations. We prove that the evaluation map is both a groupoid homotopy equivalence and a groupoid fibration.

Organization

In Section 2 we present some basic definitions and constructions for topological groupoids. We define translation groupoids and introduce the bicategory of translation groupoids resulting from inverting the essential equivalences. Section 3 introduces the model for orbifolds as groupoids that gives the setting for the construction of the path groupoid in the next section. Section 4 is devoted to the construction of the free path groupoid. We give here an explicit equivalence between all models for the path groupoid. We prove that this construction is functorial and invariant under Morita equivalence. Section 5 provides a detailed description of the based path and loop groupoids and describes some examples. Section 6 concerns the characterization of the homotopy between generalized maps. In Section 7 we provide a definition of groupoid fibration and prove that the evaluation morphism is a groupoid fibration.

2 Context

2.1 Topological groupoids

A *topological groupoid* \mathcal{G} is a groupoid object in the category Top of topological spaces and continuous maps. Our notation for groupoids is that G_0 is the space of objects and G_1 is the space of arrows, with source and target maps $s, t : G_1 \rightarrow G_0$, multiplication $m : G_1 \times_{G_0} G_1 \rightarrow G_1$, inversion $i : G_1 \rightarrow G_1$, and object inclusion $u : G_0 \hookrightarrow G_1$.

The set of arrows from x to y is denoted by $G(x, y) = \{g \in G_1 \mid s(g) = x \text{ and } t(g) = y\}$. The set of arrows from x to itself, $G(x, x)$, is a group called the *isotropy* group of \mathcal{G} at x and denoted by G_x .

A *strict morphism* $\phi : \mathcal{K} \rightarrow \mathcal{G}$ of groupoids is a functor given by two continuous maps $\phi : K_1 \rightarrow G_1$ and $\phi : K_0 \rightarrow G_0$ that together commute with all the structure maps of the groupoids \mathcal{K} and \mathcal{G} .

A *natural transformation* $T : \phi \Rightarrow \psi$ between two morphisms $\phi, \psi : \mathcal{K} \rightarrow \mathcal{G}$ is a continuous map $T : K_0 \rightarrow G_1$ with $T(x) : \phi(x) \rightarrow \psi(x)$ such that for any arrow $h : x \rightarrow y$ in K_1 , the identity $\psi(h)T(x) = T(y)\phi(h)$ holds. Since we are in a topological groupoid and inversion is continuous, we also have a natural transformation $T^{-1} : \psi \Rightarrow \phi$ and write $\phi \sim_T \psi$.

Topological groupoids, strict morphisms and natural transformations form a 2–category, which we denote by TopG .

A strict morphism $\epsilon : \mathcal{K} \rightarrow \mathcal{G}$ of topological groupoids is an *essential equivalence* if:

- (i) ϵ is essentially surjective in the sense that

$$s\pi_1 : G_1 \times_{G_0}^t K_0 \rightarrow G_0$$

is an open surjection where $G_1 \times_{G_0}^t K_0$ is the pullback along the target $t : G_1 \rightarrow G_0$.

- (ii) ϵ is fully faithful in the sense that K_1 is the pullback of topological spaces

$$\begin{array}{ccc} K_1 & \xrightarrow{\epsilon} & G_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ K_0 \times K_0 & \xrightarrow{\epsilon \times \epsilon} & G_0 \times G_0 \end{array}$$

Note that if there exists a functor $\delta : \mathcal{G} \rightarrow \mathcal{K}$ with natural transformations $\eta : \text{id}_{\mathcal{G}} \Rightarrow \epsilon \circ \delta$ and $\nu : \delta \circ \epsilon \Rightarrow \text{id}_{\mathcal{K}}$ in TopG , the functor ϵ is essentially surjective — indeed, $s\pi_1$ has

a section defined by $(\eta_x, \delta(x)): G_0 \rightarrow G_1 \times_{G_0}^t K_0$, which implies that it is open and surjective — and ϵ is fully faithful because the map $K_1 \rightarrow \mathcal{K}_0 \times \mathcal{K}_0 \times_{\mathcal{G}_0 \times \mathcal{G}_0} \mathcal{G}_1$ has an inverse defined by $(x, y, h) \rightarrow v_y \circ \delta(h) \circ v_x^{-1}$.

An essential equivalence $\epsilon: \mathcal{K} \rightarrow \mathcal{G}$ does not generally have an inverse functor $\delta: \mathcal{G} \rightarrow \mathcal{K}$ such that $\epsilon \circ \delta \sim_T \text{id}_{\mathcal{G}}$ and $\delta \circ \epsilon \sim_{T'} \text{id}_{\mathcal{K}}$ in TopG. The functor δ exists by the axiom of choice but in general it is not *continuous*.

Definition 2.1 Let $\psi: \mathcal{K} \rightarrow \mathcal{G}$ and $\phi: \mathcal{L} \rightarrow \mathcal{G}$ be strict morphisms. The *groupoid pullback* $\mathcal{P} = \mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is the topological groupoid whose space of objects is

$$P_0 = K_0 \times_{G_0}^t G_1 \times_{G_0}^s L_0$$

and space of arrows is $P_1 = K_1 \times_{G_0}^t G_1 \times_{G_0}^s L_1$. Source and target maps are given by $s(k, g, l) = (s(k), \psi(k)^{-1}g\phi(l), s(l))$ and $t(k, g, l) = (t(k), g, t(l))$. There is a square of morphisms and a natural transformation T that makes the diagram

$$\begin{array}{ccc} \mathcal{K} \times_{\mathcal{G}} \mathcal{L} & \xrightarrow{\pi_1} & \mathcal{K} \\ \pi_2 \downarrow & \sim_T & \downarrow \psi \\ \mathcal{L} & \xrightarrow{\phi} & \mathcal{G} \end{array}$$

commutative and is universal with this property.

Definition 2.2 The groupoids \mathcal{K} and \mathcal{G} are *Morita equivalent* if there exists a groupoid \mathcal{L} and a span

$$\mathcal{K} \xleftarrow{\sigma} \mathcal{L} \xrightarrow{\epsilon} \mathcal{G},$$

where ϵ and σ are essential equivalences. We write $\mathcal{G} \sim_M \mathcal{K}$.

The proof that a Morita equivalence is an equivalence relation is based on the groupoid pullback defined above.

A *generalized map* (ϵ, ϕ) from \mathcal{K} to \mathcal{G} is a span $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ such that ϵ is an essential equivalence. Two generalized maps $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ are *equivalent* if there exists a diagram

$$\begin{array}{ccccc} & & \mathcal{J} & & \\ & \epsilon \swarrow & \uparrow u & \searrow \phi & \\ \mathcal{K} & & \mathcal{L} & & \mathcal{G} \\ & \nwarrow \sim_T & \downarrow v & \nearrow \sim_{T'} & \\ & \epsilon' \swarrow & \mathcal{J}' & \searrow \phi' & \end{array}$$

which is commutative up to natural transformations and where \mathcal{L} is a topological groupoid, and u and v are essential equivalences.

2.2 The Morita bicategory of topological groupoids MTopG

Consider the class of arrows E given by the essential equivalences in the 2–category TopG. It was proven by Pronk in [13; 14] that E satisfies the conditions to admit a bicalculus of fractions. The bicategory of fractions TopG(E^{-1}) obtained by formally inverting the essential equivalences is what we call the *Morita bicategory of topological groupoids* and we denote by MTopG.

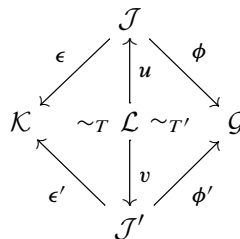
The explicit description of the bicategory MTopG is as follows:

- Objects are topological groupoids \mathcal{G} .
- A 1–morphism from \mathcal{K} to \mathcal{G} is a *generalized map*

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

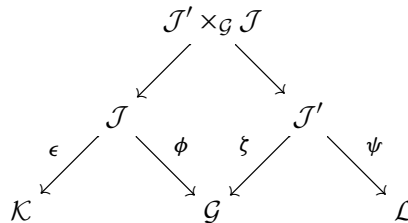
such that ϵ is an essential equivalence.

- A 2–morphism from $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ to $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ is given by a class of diagrams



where \mathcal{L} is a topological groupoid, and u and v are essential equivalences.

The horizontal composition of generalized maps $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{G} \xleftarrow{\xi} \mathcal{J}' \xrightarrow{\psi} \mathcal{L}$ is given by the diagram



where $\mathcal{J}' \times_{\mathcal{G}} \mathcal{J}$ is the groupoid pullback. Note that this composition is associative only up to a 2–morphism.

2.3 Translation groupoids

Let G be a topological group with a continuous left action on a topological space X . Then the *translation groupoid* $G \ltimes X$ is defined by:

- The space of objects is X itself, and the space of arrows is the Cartesian product $G \times X$.
- The source $s: G \times X \rightarrow X$ is the second projection, and the target $t: G \times X \rightarrow X$ is given by the action. Then (g, x) is an arrow $x \rightarrow gx$.
- The other structure maps are defined by the unit $u(x) = (e, x)$, where e is the identity element in G , and $(h, gx) \circ (g, x) = (h \star g, x)$ where \star is the group multiplication.

Example 2.3 These examples will appear later on in our applications.

- (1) **Unit groupoid** Consider the groupoid $e \ltimes X$ given by the action of the trivial group e on the topological space X . This is a topological groupoid whose arrows are all units. In this way, any topological space can be considered as a groupoid.
- (2) **Multiplication groupoid** Let H be a subgroup of a topological group G . Consider the translation groupoid $H \ltimes G$ where H acts by multiplication on G .
- (3) **Conjugation groupoid** Let H be a subgroup of a topological group G . Consider the translation groupoid $H \ltimes G$ where H acts by conjugation on G .
- (4) **Point groupoid** Let G be a topological group. Let \bullet be a point. Consider the groupoid $G \ltimes \bullet$ where G acts trivially on the point. This is a topological groupoid with exactly one object, \bullet , and G is the space of arrows in which the maps s and t coincide. We call $G \ltimes \bullet$ the point groupoid associated to G . In this way any group can be considered as a groupoid.

We will denote by $\mathbf{1}$ the *trivial groupoid* with one object and one arrow; that is, $\mathbf{1} = e \ltimes \bullet$, the unit groupoid over a point or a point groupoid associated to the trivial group.

An *equivariant map* $G \ltimes X \rightarrow K \ltimes Y$ between translation groupoids consists of a pair $\varphi \ltimes f$, where $\varphi: G \rightarrow K$ is a group homomorphism and $f: X \rightarrow Y$ satisfies $f(gx) = \varphi(g)f(x)$ for $g \in G$ and $x \in X$.

Translation groupoids, equivariant maps and natural transformations form a 2–category that we denote by TrG .

2.4 The Morita bicategory of translation groupoids MTrG

We construct now a subcategory MTrG of the Morita bicategory of topological groupoids MTopG where the objects are strictly the translation groupoids and the maps are equivariant ones.

Proposition 2.4 [14] *Let $\psi : G \times X \rightarrow L \times Z$ and $\phi : H \times Y \rightarrow L \times Z$ be equivariant maps. The fiber product \mathcal{K}*

$$\begin{array}{ccc}
 \mathcal{K} & \xrightarrow{\pi_1 \times f} & G \times X \\
 \pi_2 \times g \downarrow & & \downarrow \psi \\
 H \times Y & \xrightarrow{\phi} & L \times Z
 \end{array}$$

is again a translation groupoid. Moreover, its structure group is $G \times H$, $\mathcal{K} = (G \times H) \times P$ and the first components of the equivariant maps $\pi_1 \times f$ and $\pi_2 \times g$ are the group projections $\pi_1 : G \times H \rightarrow G$ and $\pi_2 : G \times H \rightarrow H$.

An *equivariant essential equivalence* is an equivariant map $\xi \times \epsilon$ which is an essential equivalence.

Consider the bicategory whose

- objects are translation groupoids $G \times X$;
- 1–morphisms from $G \times X$ to $K \times Y$ are *equivariant generalized maps*

$$G \times X \xleftarrow{\xi \times \epsilon} L \times Z \xrightarrow{\varphi \times f} K \times Y$$

such that $\xi \times \epsilon$ is an equivariant essential equivalence;

- a 2–morphism \Rightarrow from the equivariant generalized map $G \times X \xleftarrow{\xi \times \epsilon} L \times Z \xrightarrow{\varphi \times f} K \times Y$ to $G \times X \xleftarrow{\xi' \times \epsilon'} L' \times Z' \xrightarrow{\varphi' \times f'} K \times Y$ is given by a class of diagrams

$$\begin{array}{ccccc}
 & & L \times Z & & \\
 & \swarrow \xi \times \epsilon & \uparrow u & \searrow \varphi \times f & \\
 G \times X & \sim_T & R \times U & \sim_{T'} & K \times Y \\
 & \swarrow \xi' \times \epsilon' & \downarrow v & \searrow \varphi' \times f' & \\
 & & L' \times Z' & &
 \end{array}$$

where $R \times U$ is a translation groupoid, and u and v are equivariant essential equivalences.

Translation groupoids, equivariant generalized maps and diagrams as above form the *Morita bicategory of translation groupoids*, which we denote by MTrG.

3 Orbifolds as groupoids

We recall now the description of orbifolds as groupoids due to Moerdijk and Pronk [9; 13]. Orbifolds were first introduced by Satake [16] as a generalization of a manifold defined in terms of local quotients. The groupoid approach provides a global language to reformulate the notion of orbifold.

A groupoid \mathcal{G} is *proper* if $(s, t): G_1 \rightarrow G_0 \times G_0$ is a proper map and it is a *foliation* groupoid if each isotropy group is discrete.

Definition 3.1 An *orbifold* groupoid is a proper foliation groupoid.

Given an orbifold groupoid \mathcal{G} , its orbit space $|\mathcal{G}|$ is a locally compact Hausdorff space. Given an arbitrary locally compact Hausdorff space X we can equip it with an orbifold structure as follows:

Definition 3.2 An *orbifold structure* on a locally compact Hausdorff space X is given by an orbifold groupoid \mathcal{G} and a homeomorphism $h: |\mathcal{G}| \rightarrow X$.

If $\epsilon: \mathcal{H} \rightarrow \mathcal{G}$ is an essential equivalence and $|\epsilon|: |\mathcal{H}| \rightarrow |\mathcal{G}|$ is the induced homeomorphism between orbit spaces, we say that the composition $h \circ |\epsilon|: |\mathcal{H}| \rightarrow X$ defines an *equivalent* orbifold structure.

Definition 3.3 An *orbifold* \mathcal{X} is a space X equipped with an equivalence class of orbifold structures. A specific such structure, given by \mathcal{G} and $h: |\mathcal{G}| \rightarrow X$, is a *presentation* of the orbifold \mathcal{X} .

If two groupoids are Morita equivalent, then they define the same orbifold. Therefore any structure or invariant for orbifolds, if defined through groupoids, should be invariant under Morita equivalence.

Definition 3.4 An *orbifold map* $f: \mathcal{Y} \rightarrow \mathcal{X}$ is given by an equivalence class of generalized maps (ϵ, ϕ) from \mathcal{K} to \mathcal{G} between presentations of the orbifolds such that the following diagram commutes:

$$\begin{array}{ccc} |\mathcal{K}| & \xrightarrow{|\phi||\epsilon|^{-1}} & |\mathcal{G}| \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

A specific such generalized map (ϵ, ϕ) is called a *presentation* of the orbifold map f .

We can obtain an orbifold by considering the action of a compact group G acting on a space X with finite stabilizers. All orbifolds can be described in this way [12].

The orbifold \mathcal{X} is *developable* if it is presented by a groupoid Morita equivalent to a translation groupoid $G \ltimes X$ with G a discrete group acting properly on X .

4 Path groupoid

From now on, we will focus on developable orbifolds and G will be a discrete group acting properly on X . In this context, we will show that in the bicategory of topological groupoids any path in $G \ltimes X$

$$I \leftarrow \mathcal{I} \rightarrow G \ltimes X$$

is equivalent to a strict map

$$I \rightarrow G \ltimes X$$

where I is the unit groupoid $e \ltimes I$, $I = [0, 1]$ and \mathcal{I} is any topological groupoid.

4.1 Generalized paths

A *path* in the groupoid $G \ltimes X$ in the Morita bicategory of topological groupoids is a generalized map (δ, β) from the unit groupoid I to $G \ltimes X$. That is, a span

$$I \xleftarrow{\delta} \mathcal{I} \xrightarrow{\beta} G \ltimes X.$$

Since $I \xleftarrow{\delta} \mathcal{I}$ is an essential equivalence, we can use groupoid atlases [15; 17] to see that the equivalence class $[I \xleftarrow{\delta} \mathcal{I} \xrightarrow{\beta} G \ltimes X]$ has a representative of the form

$$I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \ltimes X,$$

where I_{S_n} is the groupoid associated to a subdivision

$$S_n = \{0 = r_0 \leq r_1 < \dots < r_{n-1} \leq r_n = 1\}$$

of the interval $I = [0, 1]$ as explained below.

The space of objects of the groupoid I_{S_n} is the disjoint union

$$\bigsqcup_{i=1}^n I_i,$$

where I_i is a small open neighborhood of $[r_{i-1}, r_i]$ and (r, i) denotes an element r in the connected component I_i .

The space of arrows of I_{S_n} is given by the disjoint union

$$\bigsqcup_{i=1}^n I_i \bigsqcup_{i=1}^{n-1} (\tilde{I}_i \sqcup \tilde{I}_i),$$

where $\bigsqcup_{i=1}^n I_i$ is the set of unit arrows, $\tilde{I}_i = I_i \cap I_{i+1}$ and another copy \tilde{I}_i was added for inverse arrows. For each point r_i in the subdivision S_n , \tilde{I}_i is an open neighborhood of r_i . Two arrows were added for each point (r, i) in the interval \tilde{I}_i : \tilde{r}_i and its inverse arrow such that the source of \tilde{r}_i is (r, i) and its target is $(r, i + 1)$.

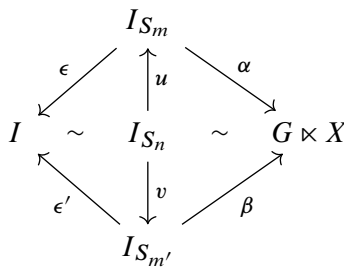
Definition 4.1 A *generalized path* in the groupoid $G \times X$ is a generalized map $I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \times X$ such that:

- (1) $\epsilon: I_{S_n} \rightarrow I$ on objects is the inclusion in each connected component, $\epsilon(r, i) = r$ and on arrows it sends all arrows to identity arrows, $\epsilon(\tilde{r}_i) = \text{id}_r$.
- (2) $\alpha: I_{S_n} \rightarrow G \times X$ on objects is given by a map $\alpha_i: I_i \rightarrow X$ in each connected component and on arrows is given by $\alpha(\tilde{r}_i) = (k_i, \alpha_i(r))$ satisfying the condition $k_i \alpha_i(r) = \alpha_{i+1}(r)$ for all $r \in \tilde{I}_i$.

We denote by $\text{Map}(I_{S_n}, G \times X)$ this space of maps from I_{S_n} to $G \times X$ with the compact open topology.

4.1.1 Equivalence of generalized paths We will establish now an equivalence relation between the generalized maps defining our generalized paths which will allow us to give a groupoid structure to the space of generalized paths.

Definition 4.2 Two generalized paths $I \xleftarrow{\epsilon} I_{S_m} \xrightarrow{\alpha} G \times X$ and $I \xleftarrow{\epsilon'} I_{S_{m'}} \xrightarrow{\beta} G \times X$ are equivalent if there exist a subdivision S_n and essential equivalences u and v such that the following diagram commutes up to natural transformations:



Since G is discrete, the condition $\alpha u \sim \beta v$ guarantees the existence of a natural transformation $T : \bigsqcup_{i=1}^n I_i \rightarrow G \times X$ such that $T(r, i) = (g_i, \alpha_i(r))$ with $\beta_i(r) = g_i \alpha_i(t)$. By naturality of the transformation, the diagram

$$\begin{array}{ccc} \alpha_i(r, i) & \xrightarrow{g_i} & \beta_i(r, i) \\ k_i \downarrow & & \downarrow k'_i \\ \alpha_{i+1}(r, i) & \xrightarrow{g_{i+1}} & \beta_{i+1}(r, i) \end{array}$$

commutes for all $r \in \tilde{I}_i$. Therefore $k'_i = g_{i+1} k_i g_i^{-1}$ for all $i = 1, \dots, n - 1$.

Remark 4.3 Two generalized paths are equivalent if there exists a common subdivision S_n and $g_i \in G$ such that $\beta_i(r) = g_i \alpha_i(r)$ for all $i = 1, \dots, n$ and $k'_i = g_{i+1} k_i g_i^{-1}$ for all $i = 1, \dots, n - 1$.

Then, we have a translation groupoid $G^n \times \text{Map}(I_{S_n}, G \times X)$ given by this action of G^n on the space $\text{Map}(I_{S_n}, G \times X)$. Source and target are given by

$$s((g_1, \dots, g_n), (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})) = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$$

and

$$\begin{aligned} t((g_1, \dots, g_n), (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})) \\ = (g_1 \alpha_1, \dots, g_n \alpha_n, g_2 k_1 g_1^{-1}, \dots, g_n k_{n-1} g_{n-1}^{-1}). \end{aligned}$$

4.1.2 Colimit construction In order to account for all possible subdivisions, we will consider the colimit of the groupoids $G^n \times \text{Map}(I_{S_n}, G \times X)$ over a partially ordered set that we describe next.

We define the category \mathcal{C}_I as the category with objects the ordered tuples

$$S_n = \{0 = r_0 \leq r_1 \leq \dots \leq r_n = 1\}$$

with an open cover of $I = [0, 1]$ given by connected intervals $\{I_i \mid 1 \leq i \leq n\}$. We require that:

- (1) $[r_{i-1}, r_i] \subseteq I_i$ and $I_i \cap \{r_0, r_1, \dots, r_n\} = \{r_{i-1}, r_i\}$, which is one point if $r_{i-1} = r_i$ and two points if $r_{i-1} < r_i$.
- (2) (a) If $r_{k-2} < r_{k-1} = r_k = \dots = r_l < r_{l+1}$ then we require that

$$I_k = I_{k+1} = \dots = I_l \subseteq I_{k-1} \cap I_{l+1}.$$

(b) If $0 = r_0 = r_1 = \dots = r_k < r_{k+1}$ then we require that

$$I_1 = I_2 = \dots = I_k \subseteq I_{k+1}.$$

(c) If $r_{k-1} < r_k = r_{k+1} = \dots = r_n = 1$ then we require that

$$I_{k+1} = I_{k+2} = \dots = I_n \subseteq I_k.$$

We have a morphism from $(\{r_0 \leq r_1 \leq \dots \leq r_n\}, \{I_i\})$ to $(\{t_0 \leq t_1 \leq \dots \leq t_m\}, \{\tilde{I}_j\})$ if:

(I) $\{r_0, r_1, \dots, r_n\} \supseteq \{t_0, t_1, \dots, t_m\}$.

(II) The multiplicity of repeated elements decreases; ie for every i ,

$$|\{j \mid r_j = r_i\}| \geq |\{j \mid t_j = r_i\}|.$$

(III) The open cover $\{I_i\}$ is a refinement of the open cover $\{\tilde{I}_j\}$ in the following way:

(a) For each closed interval $[r_{i-1}, r_i]$ with nonempty interior there is a unique $[t_{j-1}, t_j]$ with $[r_{i-1}, r_i] \subseteq [t_{j-1}, t_j]$ and we have

$$\begin{array}{ccc} I_i & \xrightarrow{\subseteq} & \tilde{I}_j \\ \subseteq \uparrow & & \subseteq \uparrow \\ [r_{i-1}, r_i] & \xrightarrow{\subseteq} & [t_{j-1}, t_j] \end{array}$$

(b) If there is a repeated element in the $\{t_0 \leq t_1 \leq \dots \leq t_m\}$, $t_{j-1} = t_j$, it is also a repeated element of $\{r_0 \leq r_1 \leq \dots \leq r_n\}$, $r_{i-1} = r_i$. We require $I_i \subseteq \tilde{I}_j$.

The morphisms are generated (as a category) by the set of morphisms:

(1) Eliminating a point from the subdivision $\{0 = r_0 \leq r_1 \leq \dots \leq r_i \leq \dots \leq r_n = 1\}$:

$$d_i: (\{r_0 \leq \dots \leq r_i \leq \dots \leq r_n\}, \{I_i\}) \rightarrow (\{r_0 \leq \dots \leq \hat{r}_i \leq \dots \leq r_n\}, \{\tilde{I}_j\}),$$

where d_i drops the i^{th} element and concatenates the consecutive intervals I_i and I_{i+1} , ie $\tilde{I}_j = I_j$ for $j = 0, \dots, i - 1$, $\tilde{I}_i = I_i \cup I_{i+1}$ and $\tilde{I}_j = I_j$ for $j = i + 1, \dots, n$.

(2) Enlarging the intervals without changing the points of the subdivision given by $\{0 = r_0 \leq r_1 \leq \dots \leq r_n = 1\}$:

$$u: (\{r_0 \leq \dots \leq r_n\}, \{I_i\}) \rightarrow (\{r_0 \leq \dots \leq r_n\}, \{\tilde{I}_i\})$$

when $I_i \subseteq \tilde{I}_i$.

We call \mathcal{C}_I the category of subdivisions of I which is a cofiltered category, which boils down to the fact that for two subdivisions there is a common refinement.

For every morphism, there is a continuous map given by concatenation and inclusion

$$\bigsqcup_i I_i \rightarrow \bigsqcup_j \tilde{I}_j.$$

To the morphism $d_i: S_n \rightarrow S_{n-1}$, we assign the functor $d_{i*}: I_{S_n} \rightarrow I_{S_{n-1}}$ that on objects concatenates $I_i \cup I_{i+1}$ and on morphisms sends \tilde{r}_i and its inverse arrow \tilde{r}'_i to the identity arrow on (r, i) . Similarly, for $u: S_n \rightarrow S_n$, there is a functor $u_*: I_{S_n} \rightarrow I_{S_n}$ given by inclusion at the level of objects and morphisms. This gives a functor from $\mathcal{C}_I \rightarrow \text{Gpd}$. We can obtain a contravariant functor ψ from $\mathcal{C}_I^{\text{op}}$ to topological spaces that on objects sends S_n to $\text{Map}(I_{S_n}, G \times X)$ and on morphisms sends $d_i: S_n \rightarrow S_{n-1}$ to the morphism $d_i^*: \text{Map}(I_{S_{n-1}}, G \times X) \rightarrow \text{Map}(I_{S_n}, G \times X)$ given by taking $\alpha \in \text{Map}(I_{S_{n-1}}, G \times X)$ represented by $(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2})$ and sending it to $(\alpha_1, \dots, \alpha_i|_{I_i}, \alpha_i|_{I_{i+1}}, \dots, \alpha_{n-1}, k_1, \dots, k_{i-1}, \text{id}, k_i, k_{i+1}, \dots, k_{n-2})$, ie taking $\alpha_i: I_i \cup I_{i+1} \rightarrow X$ to the restrictions to I_i and I_{i+1} . Similarly,

$$u^*: \text{Map}(I_{S_n}, G \times X) \rightarrow \text{Map}(I_{S_n}, G \times X)$$

is just restriction of all the paths: taking $\alpha \in \text{Map}(I_{S_n}, G \times X)$ represented by $(\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ and sending it to $(\alpha_1|_{I_1}, \dots, \alpha_n|_{I_n}, k_1, \dots, k_{n-1})$.

We have an action of G^n on the space $\text{Map}(I_{S_n}, G \times X)$ given by

$$\begin{aligned} (g_1, \dots, g_n) \cdot (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \\ = (g_1\alpha_1, \dots, g_n\alpha_n, g_2k_1g_1^{-1}, \dots, g_nk_{n-1}g_{n-1}^{-1}). \end{aligned}$$

The map $d_i^*: \text{Map}(I_{S_{n-1}}, G \times X) \rightarrow \text{Map}(I_{S_n}, G \times X)$ is equivariant with respect to the map $\sigma_i: G^{n-1} \rightarrow G^n$ given by $\sigma_i(g_1, \dots, g_{n-1}) = (g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$. This means that

$$\begin{aligned} \sigma_i(g_1, \dots, g_{n-1}) \cdot d_i^*(\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2}) \\ = d_i^*((g_1, \dots, g_{n-1}) \cdot (\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2})) \end{aligned}$$

This is because $(g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$ acting on

$$(\alpha_1, \dots, \alpha_i|_{I_i}, \alpha_i|_{I_{i+1}}, \dots, \alpha_{n-1}, k_1, \dots, k_{i-1}, \text{id}, k_i, k_{i+1}, \dots, k_{n-2})$$

is equal in the first part to

$$(g_1\alpha_1, \dots, g_i\alpha_i|_{I_i}, g_i\alpha_i|_{I_{i+1}}, \dots, g_{n-1}\alpha_{n-1})$$

and in the second part to

$$(g_2 k_1 g_1^{-1}, \dots, g_i k_{i-1} g_{i-1}^{-1}, g_i \text{id} g_i^{-1}, g_{i+1} k_i g_i^{-1}, \dots, g_{n-1} k_{n-2} g_{n-2}^{-1}),$$

which is

$$(g_2 k_1 g_1^{-1}, \dots, g_i k_{i-1} g_{i-1}^{-1}, \text{id}, g_{i+1} k_i g_i^{-1}, \dots, g_{n-1} k_{n-2} g_{n-2}^{-1}).$$

This is precisely

$$d_i^*((g_1, \dots, g_{n-1}) \cdot (\alpha_1, \dots, \alpha_{n-1}, k_1, \dots, k_{n-2})).$$

Similarly the map $u^* : \text{Map}(I_{S_n}, G \times X) \rightarrow \text{Map}(I_{S_n}, G \times X)$ is equivariant with respect to the identity map $G^n \rightarrow G^n$.

Therefore we have a contravariant functor from \mathcal{C}_I to the category of translation groupoids that on objects sends S_n to $G^n \times \text{Map}(I_{S_n}, G \times X)$ and on morphisms sends $d_i : S_n \rightarrow S_{n-1}$ to the functor (d_i^*, σ_i) and $u : S_n \rightarrow S_n$ to the functor (u^*, id) ; formally we have a (covariant) functor $\Phi : \mathcal{C}_I^{\text{op}} \rightarrow \text{TrG}$.

We consider now the (filtered) colimit of Φ ,

$$P = \text{colim}_{\mathcal{C}_I^{\text{op}}} \Phi$$

given by an object $P \in \text{TrG}$ together with morphisms from $\text{Map}(I_{S_n}, G \times X)$ for each S_n such that for each morphism the following diagrams commute:

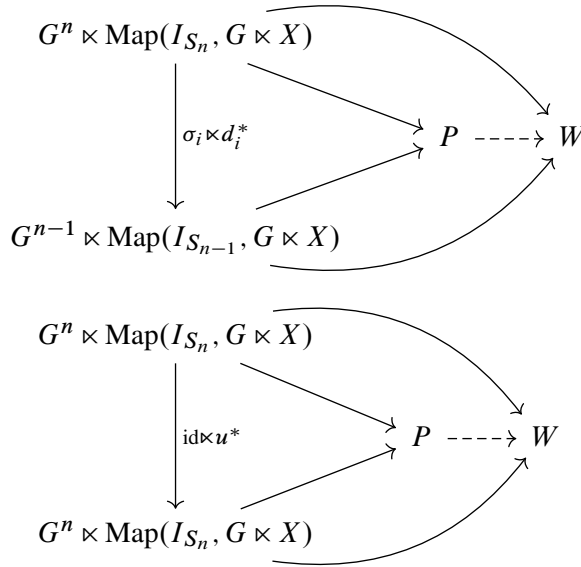
For d_i :

$$\begin{array}{ccc} G^n \times \text{Map}(I_{S_n}, G \times X) & & \\ \downarrow \sigma_i \times d_i^* & \searrow & \\ & & P \\ G^{n-1} \times \text{Map}(I_{S_{n-1}}, G \times X) & \nearrow & \end{array}$$

For u :

$$\begin{array}{ccc} G^n \times \text{Map}(I_{S_n}, G \times X) & & \\ \downarrow \text{id} \times u^* & \searrow & \\ & & P \\ G^n \times \text{Map}(I_{S_n}, G \times X) & \nearrow & \end{array}$$

Moreover, $P = \text{colim } \Phi$ has the following universal property. Given another translation groupoid W with functors from $G^n \times \text{Map}(I_{S_n}, G \times X)$ that are compatible, such functors factor uniquely through the colimit P as shown in the diagrams



Definition 4.4 The path groupoid $P(G \times X)$ of the translation groupoid $G \times X$ is

$$P(G \times X) = \text{colim}_{\mathcal{C}_I^{\text{op}}} \Phi,$$

where $\Phi : \mathcal{C}_I^{\text{op}} \rightarrow \text{TrG}$ is as above.

We are ready now to give an explicit construction of the groupoid $P = P(G \times X)$ by using the constructions of colimits in the category of topological spaces Top and in the category of groups Grp .

The colimit of the contravariant functor $\psi : \mathcal{C}_I^{\text{op}} \rightarrow \text{Top}$ is a topological space $M = \text{colim } \psi$ such that

$$M = \left(\coprod_{\mathcal{C}_I} \text{Map}(I_{S_n}, G \times X) \right) / \sim,$$

where \sim is the equivalence relation generated by $\alpha \sim d_i^*(\alpha)$ for all S_n and $d_i : S_n \rightarrow S_{n-1}$ and $\alpha \sim u^*(\alpha)$ for all S_n and $u : S_n \rightarrow S_n$.

This topological space $M = \text{colim } \psi$ will be the space of objects of the path groupoid P . To construct the space of arrows of the path groupoid, we consider now a colimit in the category of groups.

Consider the functor $\varphi: \mathcal{C}_I^{\text{op}} \rightarrow \text{Grp}$ which sends S_n to G^n and on morphisms sends $u: S_n \rightarrow S_n$ to the identity $G^n \rightarrow G^n$ and $d_i: S_n \rightarrow S_{n-1}$ to the morphism $\sigma_i: G^{n-1} \rightarrow G^n$ given by $\sigma_i(g_1, \dots, g_{n-1}) = (g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$.

The colimit of φ is a group $H = \text{colim } \varphi$ such that

$$H = \left(\coprod_{\mathcal{C}_I} G^n \right) / \sim$$

where \sim is generated by $(g_1, \dots, g_{n-1}) \sim (g_1, \dots, g_i, g_i, g_{i+1}, \dots, g_{n-1})$. This group H is discrete and acts on the topological space M constructed above.

We can describe now explicitly the object and arrow spaces of the path groupoid $P = P(G \times X)$ in TrG :

$$P_0 = M = \text{colim } \psi = \coprod_{\mathcal{C}_I} \text{Map}(I_{S_n}, G \times X) / \sim$$

and

$$P_1 = H \times M = \text{colim } \varphi \times \text{colim } \psi = \left(\left(\coprod_{\mathcal{C}_I} G^n \right) / \sim \right) \times \left(\left(\coprod_{\mathcal{C}_I} \text{Map}(I_{S_n}, G \times X) \right) / \sim \right),$$

which we endow with the inductive topology.

Remark 4.5 Let G be a discrete group acting on X . The path groupoid of $G \times X$ is the translation groupoid

$$P = P(G \times X) = H \times M.$$

We will show that this path groupoid $P = \text{colim } \Phi$ described above is actually equivalent to the translation groupoid $G \times X^I$. In order to give an explicit characterization of the equivalence of categories, we will introduce some auxiliary groupoids which in turn will relate to the idea introduced in [4] of multiple G -paths.

4.2 Multiple G -paths

We will provide now another description of the path groupoid in terms of equivariant generalized maps. We will see that for each generalized path (ϵ, α) , its equivalence class $[I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \times X]$ contains a representative in MTrG of the form

$$I \xleftarrow{\delta} G \times Y \xrightarrow{\phi} G \times X,$$

where $G \times Y$ is a translation groupoid.

Given a generalized path $I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \times X$, we will construct a space $Y = Y_\alpha$ such that $G \times Y$ is Morita equivalent to I_{S_n} , and maps $\delta: G \times Y \rightarrow I$ and $\phi: G \times Y \rightarrow G \times X$ such that (δ, ϕ) is 2-isomorphic to the given G -path (ϵ, α) .

4.2.1 Construction of $G \times Y_\alpha$ Let $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$. Consider the product space

$$G \times (I_{S_n})_0 = \{(g, (r, i)) \mid g \in G, (r, i) \in I_i\}$$

and the identifications, for all $r \in \tilde{I}_i$,

$$(g, (r, i + 1)) \sim (k_i^{-1}g, (r, i)),$$

where $\alpha(\tilde{r}_i) = (k_i, \alpha_i(r))$.

Now Y_α is defined as the quotient space

$$\{[(g, (r, i))] \mid (g, (r, i)) \in G \times (I_{S_n})_0 \text{ and } (g, (r, i + 1)) \sim (k_i^{-1}g, (r, i)) \text{ for all } r \in \tilde{I}_i\}.$$

Observe that the space Y_α depends on α in the sense that it is given by the subdivision S_n and the group elements k_1, \dots, k_{n-1} , but it is independent of the actual pieces of the path $\alpha_1, \dots, \alpha_n$.

The action of G on Y_α is given by the multiplication in the group $h([(g, (r, i))]) = [gh^{-1}, (r, i)]$.

We can consider then the translation groupoid $G \times Y_\alpha$ where the source and target are given by the maps $s(h, [g, (r, i)]) = [g, (r, i)]$ and $t(h, [g, (r, i)]) = [gh^{-1}, (r, i)]$.

4.2.2 Morita equivalence $I_{S_n} \sim_M G \times Y_\alpha$ We will show now that the translation groupoid constructed above is Morita equivalent to the groupoid I_{S_n} . Let

$$\nu: I_{S_n} \rightarrow G \times Y_\alpha$$

be the morphism defined by $\nu((r, i)) = [e, (r, i)]$ on objects and $\nu(\tilde{r}_i) = (k_i, [e, (r, i)])$ on arrows for all $r \in \tilde{I}_i$. The open map ν is essentially surjective since

$$s\pi_1: G \times (I_{S_n})_0 \rightarrow Y_\alpha = (G \times (I_{S_n})_0)/\sim$$

is the quotient projection. It is also fully faithful since $(I_{S_n})_1$ is given by the pullback of the maps

$$\begin{array}{ccc} & G \times Y_\alpha & \\ & \downarrow (s,t) & \\ (I_{S_n})_0 \times (I_{S_n})_0 & \xrightarrow{\nu \times \nu} & Y_\alpha \times Y_\alpha \end{array}$$

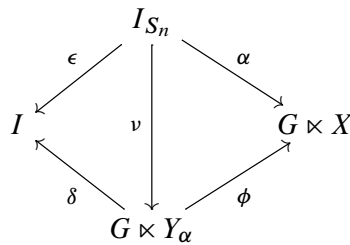
Therefore given a groupoid I_{S_n} , we can construct another groupoid Y_α for each set of elements k_1, \dots, k_{n-1} such that I_{S_n} is Morita equivalent to $G \times Y_\alpha$.

4.2.3 The 2-isomorphism $(\epsilon, \alpha) \Rightarrow (\delta, \phi)$ We will define now the maps δ and ϕ to obtain the generalized map $I \xleftarrow{\delta} G \times Y_\alpha \xrightarrow{\phi} G \times X$ being 2-isomorphic to the given generalized path (ϵ, α) .

We define $\phi([g, (r, i)]) = g^{-1}\alpha_i(r)$ on objects and $\phi(h, [g, (r, i)]) = (h, g^{-1}\alpha_i(r))$ on arrows. Moreover, the morphism ϕ is G -equivariant in the ordinary sense (the group homomorphism is the identity).

The essential equivalence $\delta: G \times Y_\alpha \rightarrow I$ is given by projection on both objects and arrows, $\delta(h, [g, (r, i)]) = r$. Both morphisms ϕ and δ are well defined and δ is open, surjective on objects and fully faithful.

The diagram



is commutative since $\phi v((r, i)) = \phi([e, (r, i)]) = \alpha_i(r_i)$ and $\phi v(\tilde{r}_i) = \phi(k_i, [e, (r, i)]) = (k_i, \alpha_i(r))$ for all $r \in \tilde{I}_i$.

Thus there is a 2-isomorphism between the generalized map $I \xleftarrow{\delta} G \times Y_\alpha \xrightarrow{\phi} G \times X$ and the generalized path $I \xleftarrow{\epsilon} I_{S_n} \xrightarrow{\alpha} G \times X$.

Observe that the identifications we have made in the quotient to obtain the space Y_α determine a gluing of the segments I_i at the different levels of $G \times (I_{S_n})_0$ to obtain copies of the entire interval $I = [0, 1]$. This gluing is determined by the group elements k_1, \dots, k_{n-1} .

To define the map ϕ from the groupoid $G \times Y_\alpha$ associated to the generalized path α , we are concatenating the different pieces α_i in these different levels by multiplying by the correct group element to obtain an honest path in X .

4.2.4 The homeomorphism $\gamma: Y_\alpha \rightarrow G \times I$ For each map $\alpha: I_{S_n} \rightarrow G \times X$, let us show now that the space Y_α we just constructed is G -equivariantly homeomorphic to

the space $G \times I$, where the action on the latter is determined by the action of G on Y_α given by $h[g, (r, i)] = [gh^{-1}, (r, i)]$. We have that the action on $G \times I$ is given by

$$G \times (G \times I) \rightarrow G \times I, \quad (h, (g, r)) = (gh^{-1}, r).$$

We define the homeomorphism $\gamma: Y_\alpha \rightarrow G \times I$ as

$$\gamma([g, (r, i)]) = ((k_{i-1} \cdots k_1)^{-1}g, r)$$

for $i = 1, \dots, n$. The morphism γ depends only on S_n and k_1, \dots, k_{n-1} and is independent on the actual paths $\alpha_1, \dots, \alpha_n$. The inverse morphism $\gamma^{-1}: G \times I \rightarrow Y_\alpha$ is given by

$$\gamma^{-1}(h, r) = [k_{i-1} \cdots k_1 h, (r, i)]$$

if $r \in I_i$. Moreover, the homeomorphism γ is G -equivariant by construction.

Definition 4.6 A multiple G -path in the groupoid $G \times X$ is a generalized map

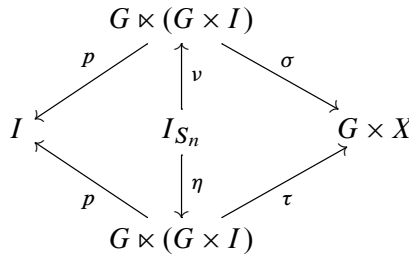
$$I \leftarrow G \times (G \times I) \xrightarrow{\sigma} G \times X,$$

where σ is a G -equivariant map in the ordinary sense.

4.2.5 Equivalence of multiple G -paths Given two multiple G -paths

$$I \leftarrow G \times (G \times I) \xrightarrow{\sigma} G \times X \quad \text{and} \quad I \leftarrow G \times (G \times I) \xrightarrow{\tau} G \times X,$$

they are equivalent if there exists a subdivision S_n and $k_1, \dots, k_{n-1} \in G$ such that the diagram



commutes up to natural transformations, where $v = v_{k_1, \dots, k_{n-1}}$ and $\eta = \eta_{k_1, \dots, k_{n-1}}$.

Since p is an essential equivalence, we have that $v \sim \eta$ and then $\sigma v \sim \tau v$. That means that there exists a natural transformation $T: (I_{S_n})_0 \rightarrow G \times X$ such that $T(r, i)$ is an arrow between $\sigma v(r, i) = \sigma((k_{i-1} \cdots k_1)^{-1}, r)$ and $\tau((k_{i-1} \cdots k_1)^{-1}, r)$. Therefore we have that the multiple G -paths are equivalent if there exists a subdivision S_n , $k_1, \dots, k_{n-1} \in G$ and $g_1, \dots, g_n \in G$ such that

$$g_i \sigma((k_{i-1} \cdots k_1)^{-1}, r) = \tau((k_{i-1} \cdots k_1)^{-1}, r) \quad \text{if } r \in I_i.$$

Since σ is equivariant,

$$g_i(k_{i-1} \cdots k_1)\sigma(e, r) = (k_{i-1} \cdots k_1)\tau(e, r) \quad \text{if } r \in I_i.$$

For $i = 1$ this means that there exists $g_1 \in G$ such that $\tau(e, r) = g_1\sigma(e, r)$. Since the interval $e \times I$ is connected, we have that $g_i = (k_{i-1} \cdots k_1)g_1(k_{i-1} \cdots k_1)^{-1}$ for all $i = 1, \dots, n$. In other words, all other g_i for $i = 2, \dots, n$ are determined by g_1 . Once that we have a group element $g_1 \in G$ that makes $\tau(e, r) = g_1\sigma(e, r)$ in the first piece of the interval, $r \in [0, r_1]$, then all the other pieces of the interval coming from the subdivision S_n will also coincide since, for all $r \in I_i$,

$$g_i(k_{i-1} \cdots k_1)\sigma(e, r) = (k_{i-1} \cdots k_1)\tau(e, r)$$

and

$$g_i = (k_{i-1} \cdots k_1)g_1(k_{i-1} \cdots k_1)^{-1}.$$

Then

$$(k_{i-1} \cdots k_1)g_1(k_{i-1} \cdots k_1)^{-1}(k_{i-1} \cdots k_1)\sigma(e, r) = (k_{i-1} \cdots k_1)\tau(e, r),$$

which implies that $g_1\sigma(e, r) = \tau(e, r)$ for all $r \in I$.

Proposition 4.7 *Two multiple G -paths σ and τ are equivalent if there exists $g \in G$ such that*

$$g\sigma(e, r) = \tau(e, r).$$

We have the group G acting now on the space of equivariant maps $G\text{Map}(G \times I, X)$. Let $P' = G \times G\text{Map}(G \times I, X)$ be the *multiple G -path groupoid*.

Since $\sigma(g, r) = g\sigma(e, r)$, we observe that a multiple G -path is determined by the honest path $\beta: I \rightarrow X$ given by $\beta(r) = \sigma(e, r)$. Conversely, any path $\beta: I \rightarrow X$ can be made into a multiple G -path by putting $\sigma(g, r) = g\beta(r)$. Consider the translation groupoid of honest paths, given by the obvious action of G on X . Let $P'' = G \times X^I$, where $X^I = \text{Map}(I, X)$.

We will prove next that all three characterizations of the path groupoid, as generalized paths, as multiple G -paths and as honest paths are equivalent.

4.3 Equivalence of the different models for path groupoids

Recall the definition of the path groupoid and the other two characterizations introduced in the previous section:

- (1) The groupoid $P = \text{colim } \varphi \times \text{colim } \psi$, where $M = \text{colim } \psi$ is the space of classes of generalized paths.
- (2) The groupoid $P' = G \times G\text{Map}(G \times I, X)$, where $G\text{Map}(G \times I, X)$ is the space of G -equivariant maps.
- (3) The groupoid $P'' = G \times X^I$, where X^I is the free path space.

4.3.1 The equivalence of categories $\chi : \text{colim } \varphi \times \text{colim } \psi \rightarrow G \times G\text{Map}(G \times I, X)$

Recall that $M = \text{colim } \psi$ is the space of classes of generalized paths, ie

$$M = \left(\coprod_{c_I} \text{Map}(I_{S_n}, G \times X) \right) / \sim,$$

where \sim is the equivalence relation generated by $\alpha \sim d_i^*(\alpha)$ for all S_n and $d_i : S_n \rightarrow S_{n-1}$ and $\alpha \sim u^*(\alpha)$ for all S_n and $u : S_n \rightarrow S_n$. We will use the same notation,

$$(\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}),$$

to denote the elements in M .

The idea is to complete each piece α_i of the generalized path

$$\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$$

to have the entire branch σ_i of a multiple G -path σ .

The functor $\chi : \text{colim } \varphi \times \text{colim } \psi \rightarrow G \times G\text{Map}(G \times I, X)$ Given a generalized path $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ for the subdivision S_n of the interval I , we can define (as in the previous section)

- (1) a space $Y_\alpha = \{[(g, (r, i))] \mid (g, (r, i)) \in G \times (I_{S_n})_0\}$ with the relation

$$(g, (r, i + 1)) \sim (k_i^{-1}g, (r, i))$$

for all $r \in \tilde{I}_i$,

- (2) a homeomorphism $\gamma_\alpha : G \times Y_\alpha \rightarrow G \times (G \times I)$,
- (3) an essential equivalence $\nu_\alpha : I_{S_n} \rightarrow G \times Y_\alpha$, and
- (4) a generalized map $I \xleftarrow{\delta} G \times Y_\alpha \xrightarrow{\phi_\alpha} G \times X$ such that $(\epsilon, \alpha) \Rightarrow (\delta, \phi_\alpha)$.

We define $\chi : \text{colim } \varphi \times M \rightarrow G \times G\text{Map}(G \times I, X)$ as $\chi(\alpha) = \phi_\alpha \gamma_\alpha^{-1}$ on objects and $\chi(g_1, \dots, g_n, \alpha) = (g_1, \phi_\alpha \gamma_\alpha^{-1})$ on arrows.

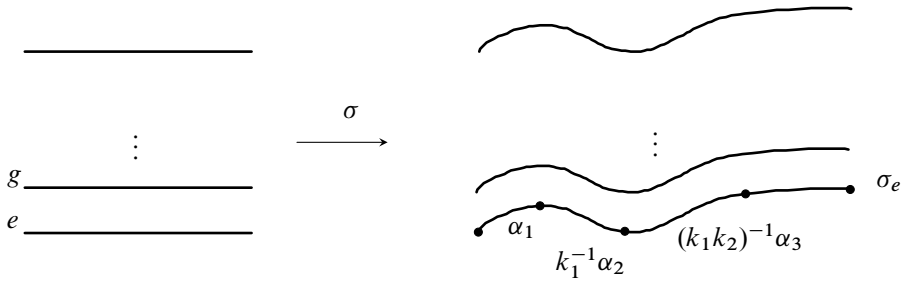


Figure 1

Then $\chi(\alpha)(g, r) = \phi_\alpha \gamma_\alpha^{-1}(g, r) = \phi_\alpha[k_{i-1} \cdots k_1 g, (r, i)] = (k_{i-1} \cdots k_1 g)^{-1} \alpha_i(r)$ if $r \in I_i$. We are sending each generalized path $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ into the multiple G -path σ given by

$$\sigma(g, r) = g^{-1}(k_{i-1} \cdots k_1)^{-1} \alpha_i(r) \quad \text{if } r \in I_i.$$

In particular, we have that the branch σ_e corresponding to the interval $e \times I$ is given by the concatenation (see also Figure 1)

$$\sigma(e, r) = \alpha_1(r) * k_1^{-1} \alpha_2(r) * (k_2 k_1)^{-1} \alpha_3(r) * \cdots * (k_{n-1} \cdots k_1)^{-1} \alpha_n(r).$$

On arrows, we send $((g_1, \dots, g_n), \alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \in \text{colim } \varphi \times \text{colim } \psi$ into the arrow (g_1, σ_α) , where σ_α is defined as before.

We will show next that χ is an equivariant map between translation groupoids where the group homomorphism is given by the projection on the first coordinate.

Let $\alpha' = (g_1 \alpha_1, \dots, g_n \alpha_n, g_2 k_1 g_1^{-1}, \dots, g_n k_{n-1} g_{n-1}^{-1})$. We have that

$$\chi(\alpha') = g_1 \chi((\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}))$$

since

$$\begin{aligned} \sigma_{\alpha'}(e, r) &= g_1 \alpha_1(r) * (g_2 k_1 g_1^{-1})^{-1} g_2 \alpha_2(r) * \cdots * (g_n k_{n-1} g_{n-1}^{-1} \cdots g_2 k_1 g_1^{-1})^{-1} g_n \alpha_n(r) \\ &= g_1 (\alpha_1(r) * k_1^{-1} \alpha_2(r) * (k_2 k_1)^{-1} \alpha_3(r) * \cdots * (k_{n-1} \cdots k_1)^{-1} \alpha_n(r)) = g_1 \sigma_\alpha(e, r). \end{aligned}$$

The functor $\chi^{-1}: G \ltimes G \text{Map}(G \times I, X) \rightarrow \text{colim } \varphi \ltimes \text{colim } \psi$ Consider the continuous functor given by $\chi^{-1}(\sigma) = \sigma|_{e \times I} \circ i_e$ on objects and $\chi^{-1}((g, \sigma)) = (g, \sigma|_{e \times I} \circ i_e)$ on arrows, where $i_e: I \rightarrow e \times I$ sends $r \in I$ to $(e, r) \in e \times I$. Recall that by our notation convention the right side means in both cases the class in the colimit. Note that the generalized path $\sigma|_{e \times I} \circ i_e$ corresponds to a subdivision S_1 with only one subinterval; that is, $\sigma|_{e \times I} \circ i_e$ is an honest path.

The functors χ and χ^{-1} are inverse up to natural transformation The composition $\chi \circ \chi^{-1}: G \times G\text{Map}(G \times I, X) \rightarrow G \times G\text{Map}(G \times I, X)$ is the identity map since the generalized map α_σ associated to σ has only one piece. On objects,

$$\chi \circ \chi^{-1}(\sigma) = \chi(\alpha_\sigma) = \sigma_{\alpha_\sigma}$$

such that $\sigma_{\alpha_\sigma}(g, r) = g^{-1}\sigma(e, r) = \sigma(g, r)$, so $\sigma_{\alpha_\sigma} = \sigma$. On arrows,

$$\chi \circ \chi^{-1}(g, \sigma) = \chi(g, \sigma_{\alpha_\sigma}) = \chi(g, \sigma) = (g, \sigma).$$

We will prove next that the composition in the other direction is equivalent by a natural transformation to the identity. We have that

$$\chi^{-1} \circ \chi: \text{colim } \varphi \times \text{colim } \psi \rightarrow \text{colim } \varphi \times \text{colim } \psi$$

sends each generalized path class $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ to the generalized path α_{σ_α} , where

$$\alpha_{\sigma_\alpha}(r) = \sigma_\alpha(e, r) = \alpha_1(r) * k_1^{-1}\alpha_2(r) * (k_2k_1)^{-1}\alpha_3(r) * \dots * (k_{n-1} \dots k_1)^{-1}\alpha_n(r),$$

and each arrow $((g_1, \dots, g_n), \alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \in \text{colim } \varphi \times \text{colim } \psi$ to the arrow $(g_1, \alpha_{\sigma_\alpha})$.

There is a natural transformation $T: \text{colim } \psi \rightarrow \text{colim } \varphi \times \text{colim } \psi$ given by

$$T(\alpha) = ((\text{id}, k_1^{-1}, (k_2k_1)^{-1}, \dots, (k_{n-1} \dots k_1)^{-1}), (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}))$$

which is an arrow between α and α_{σ_α} since

$$\begin{aligned} &(\text{id}, k_1^{-1}, (k_2k_1)^{-1}, \dots, (k_{n-1} \dots k_1)^{-1})(\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}) \\ &= ((\text{id } \alpha_1, k_1^{-1}\alpha_2, \dots, (k_{n-1} \dots k_1)^{-1}\alpha_n), (k_1^{-1}k_1, \dots, (k_{n-1} \dots k_1)^{-1}k_{n-1}(k_{n-2} \dots k_1))) \\ &\quad ((\alpha_1, k_1^{-1}\alpha_2, \dots, (k_{n-1} \dots k_1)^{-1}\alpha_n), (\text{id}, \dots, \text{id})). \end{aligned}$$

This generalized path is equal to the concatenation of the n pieces

$$\alpha_1(r) * k_1^{-1}\alpha_2(r) * (k_2k_1)^{-1}\alpha_3(r) * \dots * (k_{n-1} \dots k_1)^{-1}\alpha_n(r)$$

since the connecting arrows are all identities. Moreover, T satisfies the naturality condition and is continuous by the universal property of the colimit.

Therefore χ is an equivalence of categories between the groupoid of generalized paths and the groupoid of multiple G -paths. We will see next that the groupoid of multiple G -paths is just the free path space X^I together with G acting on it.

4.3.2 The isomorphism of categories $\xi: G \ltimes G\text{Map}(G \times I, X) \rightarrow G \ltimes X^I$ To construct this isomorphism we will use the fact that a multiple G -path σ is determined by its value at the branch σ_e corresponding to the interval $e \times I$, since σ is equivariant. We define $\xi(\sigma) = \sigma i_e \in X^I$ on objects and $\xi(g, \sigma) = (g, \sigma i_e)$ on arrows. Conversely, $\xi^{-1}(\beta) = \sigma_\beta$ where $\sigma_\beta(g, r) = g^{-1}\beta(r)$. The functor

$$\xi: G \ltimes G\text{Map}(G \times I, X) \rightarrow G \ltimes X^I$$

is an isomorphism of categories since it has a strict inverse functor, $\xi \circ \xi^{-1} = \text{id}_{G \ltimes X^I}$ and $\xi^{-1} \circ \xi = \text{id}_{G \ltimes G\text{Map}(G \times I, X)}$, satisfying that the restriction ξ and the action ξ^{-1} are both continuous.

Theorem 4.8 *All models for the path groupoid of $G \ltimes X$ are equivalent;*

$$P(G \ltimes X) = \text{colim } \varphi \ltimes \text{colim } \psi \sim G \ltimes G\text{Map}(G \times I, X) = G \ltimes X^I.$$

Remark 4.9 We can also prove that any generalized map is equivalent to a strict map in the context of translation groupoids, without using groupoids atlases. It was proven by Pronk and Scull in [14] that any generalized map

$$I \xleftarrow{\delta} \mathcal{I} \xrightarrow{\beta} G \ltimes X$$

between translation groupoids is equivalent to a generalized map

$$I \xleftarrow{\epsilon} G \ltimes Y \xrightarrow{\beta'} G \ltimes X,$$

where the middle groupoid is a translation groupoid. In the same paper, they proved that any essential equivalence between translation groupoids has to be of some prescribed form. In our case, this implies that the essential equivalence $I \xleftarrow{\epsilon} G \ltimes Y$ satisfies $e = G/K$ and $I = Y/K$ where K acts freely on Y . Hence $K = G$ and G acts freely on Y . Since G acts also properly on Y we have that $Y = G \times I$. Then any generalized map (δ, β) is equivalent to a generalized map

$$I \leftarrow G \ltimes (G \times I) \xrightarrow{\beta''} G \ltimes X.$$

Now, applying our isomorphism $\xi: P' \rightarrow P''$ to the right leg of the span, we obtain $\alpha = \xi(\beta'') \in X^I$ which gives the equivalent strict map $I \xrightarrow{\alpha} G \ltimes X$.

4.4 Functoriality and Morita invariance of the path groupoid

In this section we will see that the path groupoid is functorial and that the path groupoid is well defined up to Morita equivalence.

4.4.1 Functoriality We will show that an equivariant map between translation groupoids induces an equivariant map between the path groupoids.

For a strict equivariant map $\varphi \times f : G \times X \rightarrow H \times Y$, we have an induced map $\varphi_* \times f_* : G \times X^I \rightarrow H \times Y^I$ defined by $f_*(\alpha) = f \circ \alpha$ for all $\alpha \in X^I$ and $\varphi_* = \varphi$. We construct now an equivariant map $P(\varphi \times f) : P(G \times X) \rightarrow P(H \times Y)$ between the colimit constructions.

For every n we have induced maps

$$(\varphi \times f)_* : \text{Map}(I_{S_n}, G \times X) \rightarrow \text{Map}(I_{S_n}, H \times Y)$$

in terms of the description $\text{Map}(I_{S_n}, G \times X) = G^n \times (X^I)^n \times_{X^{n-1}} G^{n-1}$; this map corresponds just to $\varphi^n \times (f_*^{n-1} \times \varphi^n)$. By taking the colimit we obtain an equivariant map $P(\varphi \times f) : P(G \times X) \rightarrow P(H \times Y)$.

Similarly, we have an induced map

$$\varphi_* \times f_* : G \times G\text{Map}(G \times I, X) \rightarrow H \times H\text{Map}(H \times I, Y)$$

between the multiple G -path groupoids. We consider an equivariant map $(G \times I) \xrightarrow{\sigma} X$ and define $f_*(\sigma) : H \times I \rightarrow Y$ by $f_*(\sigma)(h, r) = h^{-1} f(\sigma(e, r))$.

In any of the three models the functoriality is easy to check and we have the following theorem.

Theorem 4.10 *The path groupoid of $G \times X$ is functorial for equivariant maps.*

Moreover, the equivalence of the three models for the path groupoid is natural.

Theorem 4.11 *For a strict equivariant map $\varphi \times f : G \times X \rightarrow H \times Y$ the following diagram is commutative:*

$$\begin{array}{ccc} P(G \times X) & \xrightarrow{P(\varphi \times f)} & P(H \times Y) \\ \downarrow \chi & & \downarrow \chi \\ G \times G\text{Map}(G \times I, X) & \xrightarrow{\varphi_* \times f_*} & H \times H\text{Map}(H \times I, Y) \\ \downarrow \xi & & \downarrow \xi \\ G \times X^I & \xrightarrow{\varphi_* \times f_*} & H \times Y^I \end{array}$$

4.4.2 Morita invariance We will start by proving that an essential equivalence $G \times X \rightarrow H \times Y$ induces an essential equivalence between the path groupoids,

$$P(G \times X) \rightarrow P(H \times Y).$$

This will give that for a given Morita equivalence

$$G \rtimes X \xleftarrow{\sigma} G' \rtimes X' \xrightarrow{\epsilon} H \rtimes Y,$$

where ϵ and σ are essential equivalences, we have induced essential equivalences

$$P(G \rtimes X) \xleftarrow{P(\sigma)} P(G' \rtimes X') \xrightarrow{P(\epsilon)} P(H \rtimes Y).$$

Proposition 4.12 *If $\varphi \times f : G \rtimes X \rightarrow H \rtimes Y$ is an essential equivalence, then $P(\varphi \times f)$ is an essential equivalence.*

Proof (1) $P(\varphi \times f)$ is fully faithful. We will show that $P(G \rtimes X)_1$ is the pullback of topological spaces

$$\begin{array}{ccc} P(G \rtimes X)_1 & \xrightarrow{\epsilon} & P(H \rtimes Y)_1 \\ \downarrow (s,t) & & \downarrow (s,t) \\ P(G \rtimes X)_0 \times P(G \rtimes X)_0 & \xrightarrow{\epsilon \times \epsilon} & P(H \rtimes Y)_0 \times P(H \rtimes Y)_0 \end{array}$$

Specifically we have to prove that the natural map ξ from $P(G \rtimes X)_1$ to the fibered product $P(G \rtimes X)_0 \times_{P(H \rtimes Y)_0 \times P(H \rtimes Y)_0} P(H \rtimes Y)_1$ is a homeomorphism.

Let us define the inverse map ξ^{-1} . For $\alpha, \beta \in P(G \rtimes X)_0$ and an element $\theta \in P(H \rtimes Y)_1$ with $s(\theta) = t(\theta) = \alpha$, we can assume that there is a subdivision of the interval such that α and β are represented both by elements of $\text{Map}(I_{S_n}, G \rtimes X)$ and θ by an element of H^n . Therefore we have $\alpha = (\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1})$ and $\beta = (\beta_1, \dots, \beta_n, k'_1, \dots, k'_{n-1})$ such that $f(\beta_i(r)) = h_i f(\alpha_i(r))$ for all $i = 1, \dots, n$ and $k'_i = h_{i+1} k_i h_i^{-1}$ for all $i = 1, \dots, n - 1$.

But then by fixing r and using that $\varphi \times f$ is an essential equivalence, we have a fibered product of topological spaces

$$\begin{array}{ccc} (G \rtimes X)_1 & \xrightarrow{\epsilon} & (H \rtimes Y)_1 \\ \downarrow (s,t) & & \downarrow (s,t) \\ (G \rtimes X)_0 \times (G \rtimes X)_0 & \xrightarrow{\epsilon \times \epsilon} & (H \rtimes Y)_0 \times (H \rtimes Y)_0 \end{array}$$

and therefore for every $r \in I_i$ there is $g_i^r \in G$ such that $\phi(g_i^r) = h_i$. Since G is discrete and the dependence on r is continuous, the n -tuple (g_1^r, \dots, g_n^r) actually does not depend on r and represents an element of $P(G \rtimes X)_1$.

(2) $P(\varphi \times f)$ is essentially surjective. We will show that

$$s\pi_1: P(H \times Y)_1 \times_{P(H \times Y)_0}^t P(G \times X)_0 \rightarrow P(H \times Y)_0$$

is an open surjection.

For étale groupoids, the condition that the morphism

$$s\pi_1: (H \times Y)_1 \times_{(H \times Y)_0}^t (G \times X)_0 \rightarrow (H \times Y)_0$$

is an open surjection implies that it has local sections. We will use these local sections to construct local sections of $s\pi_1: P(H \times Y)_1 \times_{P(H \times Y)_0}^t P(G \times X)_0 \rightarrow P(H \times Y)_0$.

Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a cover of Y and $s_\alpha: U_\alpha \rightarrow (H \times Y)_1 \times_{(H \times Y)_0}^t (G \times X)_0$ the local sections. Take $\gamma \in P(H \times Y)_0$ and suppose that γ is represented by an element of $\text{Map}(I_{S_n}, H \times Y)$; therefore $\gamma = (\gamma_1, \dots, \gamma_n, k_1, \dots, k_{n-1})$ with $\gamma_i: I_i \rightarrow Y$.

Given the subdivision $\{0 = r_0 \leq r_1 \leq \dots \leq r_n = 1\}$ associated to γ and with an open cover of $[0, 1]$ given by connected intervals $\{I_i \mid 1 \leq i \leq n\}$, we want to construct a refinement of the subdivision

$$\{0 = r_0 \leq s_0^1 \leq \dots \leq s_{m_1}^1 = r_1 \leq \dots \leq r_{i-1} = s_0^i \leq \dots \leq s_{m_i}^i = r_i \leq \dots \leq r_n = 1\}$$

along with connected intervals I_j^i with the property that $\gamma_i(I_j^i) \subseteq U_{\alpha_j^i}$ for some α_j^i . To construct the subdivision, first for $[r_{i-1}, r_i]$ with nonempty interior, we consider the covering $\{I_i \cap U_\alpha\}_{\alpha \in \Delta}$. By compactness of the interval $[r_{i-1}, r_i]$ we can find a partition $r_{i-1} = s_0^i < \dots < s_{m_i}^i = r_i$ such that each $\gamma_i([s_{j-1}^i, s_j^i])$ is contained in some $U_{\alpha_j^i}$. Let I_j^i be an open connected neighborhood of $[s_{j-1}^i, s_j^i]$ small enough such that $I_j^i \cap \{s_0^i, s_1^i, \dots, s_{m_i}^i\} = \{s_{j-1}^i, s_j^i\}$.

For the repeated elements $r_{i-1} = r_i$, it is a matter of just shrinking the interval I_i to get $\gamma_i(I_i) \subseteq U_{\alpha_i}$ for some α_i and to obtain an object of the category of subdivisions.

With the local sections $s_{\alpha_j^i}$ we obtain maps $\pi_2 s_{\alpha_j^i}(\gamma_i(r)): I_j^i \rightarrow X$ and functions $\pi_1 \pi_2 s_{\alpha_j^i}(\gamma_i(r)): I_j^i \rightarrow H$, since the intervals are connected and H is a discrete group, actually these functions are constant and we have elements $h_j^i \in H$ with $s\pi_1(f(\pi_2 s_{\alpha_j^i}(\gamma_i(r))), h_j^i) = \gamma_i(r)$, ie

$$f(\pi_2 s_{\alpha_j^i}(\gamma_i(r))) = h_j^i \gamma_i(r)$$

for $r \in I_j^i$.

Note that $(h_j^i)^{-1} f(\pi_2 s_{\alpha_j^i}(\gamma_i(s_j^i))) = (h_{j+1}^i)^{-1} f(\pi_2 s_{\alpha_{j+1}^i}(\gamma_i(s_j^i)))$ (both are $\gamma_i(s_j^i)$) and therefore

$$h_{j+1}^i (h_j^i)^{-1} f(\pi_2 s_{\alpha_j^i}(\gamma_i(s_j^i))) = f(\pi_2 s_{\alpha_{j+1}^i}(\gamma_i(s_j^i))).$$

Since f is full and faithful, there is a $g_j^i \in G$ with $\phi(g_j^i) = h_{j+1}^i (h_j^i)^{-1}$ such that

$$g_j^i \pi_2 s_{\alpha_j^i}(\gamma_i(s_j^i)) = \pi_2 s_{\alpha_{j+1}^i}(\gamma_i(s_j^i)).$$

Similarly at the intersection points of two consecutive paths we have $k_i \gamma_i(r) = \gamma_{i+1}(r)$ for all $r \in \tilde{I}_j^i$ and therefore

$$k_i (h_{m_i}^i)^{-1} f(\pi_2 s_{\alpha_{m_i}^i}(\gamma_i(r_i))) = k_i \gamma_i(r_i) = \gamma_{i+1}(r_i) = (h_0^{i+1})^{-1} f(\pi_2 s_{\alpha_0^{i+1}}(\gamma_{i+1}(r_i))).$$

Then

$$h_0^{i+1} k_i (h_{m_i}^i)^{-1} f(\pi_2 s_{\alpha_{m_i}^i}(\gamma_i(r_i))) = f(\pi_2 s_{\alpha_0^{i+1}}(\gamma_{i+1}(r_i))),$$

and since f is full and faithful, we have elements $g^i \in G$ with $\phi(g^i) = h_0^{i+1} k_i (h_{m_i}^i)^{-1}$ such that

$$g^i \pi_2 s_{\alpha_{m_i}^i}(\gamma_i(r_i)) = \pi_2 s_{\alpha_0^{i+1}}(\gamma_{i+1}(r_i)).$$

Therefore we have a generalized path

$$((\pi_2 s_{\alpha_j^i}(\gamma_i(r)))_{i,j}, g_1^1, g_2^1, \dots, g_{m_1}^1, g^1, g_1^2, \dots, g_{m_n}^n)$$

and elements $(h_1^1, h_2^1, \dots, h_{m_1-1}^1, \dots, h_{m_n-1}^n)$ of H . By construction,

$$(h_1^1, h_2^1, \dots, h_{m_1-1}^1, \dots, h_{m_n-1}^n)((\pi_2 s_{\alpha_j^i}(\gamma_i(r)))_{i,j}, g_1^1, g_2^1, \dots, g_{m_1}^1, g^1, g_1^2, \dots, g_{m_n}^n)$$

is

$$(\gamma_1|_{I_1^1}, \gamma_1|_{I_2^1}, \dots, \gamma_1|_{I_{m_1}^1}, \dots, \gamma_n|_{I_{m_n}^n}, \text{id}, \text{id}, \dots, k_1, \text{id}, \dots, k_n).$$

In the colimit this represents the same element as $(\gamma_1, \dots, \gamma_n, k_1, \dots, k_{n-1})$. Therefore we have constructed local sections on the set

$$\{(\gamma_1, \dots, \gamma_n, k_1, \dots, k_{n-1}) \in \text{Map}(I_{S_n}, H \times Y) \mid \gamma_i([s_{j-1}^i, s_j^i]) \subseteq U_{\alpha_j^i}\},$$

which is an open set in the compact open topology of $\text{Map}(I_{S_n}, H \times Y)$. □

Thus, we have proved that the path groupoid functor sends essential equivalences to essential equivalences and therefore the path groupoid is invariant under Morita equivalence.

Theorem 4.13 *If $G \times X \sim_M H \times Y$, then $P(G \times X) \sim_M P(H \times Y)$.*

4.5 The free loop groupoid $L(G \ltimes X)$

We use the model of the path groupoid $P'' = G \ltimes X^I$ to define the loop groupoid as the following pullback along the diagonal:

$$(1) \quad \begin{array}{ccc} & G \ltimes X^I & \\ & \downarrow \text{ev} & \\ \Delta: G \ltimes X & \longrightarrow & (G \times G) \ltimes (X \times X) \end{array}$$

Definition 4.14 The free loop groupoid $L(G \ltimes X)$ of a translation groupoid $G \ltimes X$ is

$$L(G \ltimes X) = (G \times G) \ltimes L_0,$$

where

$$L_0 = \{(\beta, h, l) \in X^I \times G \times G \mid \beta(0) = hl^{-1}\beta(1)\}$$

and the group $G \times G$ acts on L_0 by $(a, b)(\beta, h, l) = (a\beta, bha^{-1}, bla^{-1})$.

Figure 2 depicts an arrow $(a, b) \in G \times G$ from (β, h, l) to $(a\beta, bha^{-1}, bla^{-1})$.

We will show that this groupoid $(G \times G) \ltimes L_0$ is Morita equivalent to the translation groupoid $G \ltimes L$ where

$$L = \{(\alpha, g) \in X^I \times G \mid \alpha(0) = g\alpha(1)\}$$

and the action is given by $(\alpha, g) \sim (k\alpha, kgk^{-1})$. Figure 3 depicts an arrow $(k, (\alpha, g))$ between (α, g) and $(k\alpha, kgk^{-1})$.

Proposition 4.15 If $G \ltimes X$ is a topological groupoid, then the loop groupoid $L(G \ltimes X)$ is Morita equivalent to $G \ltimes L$, where L and the action are defined above.

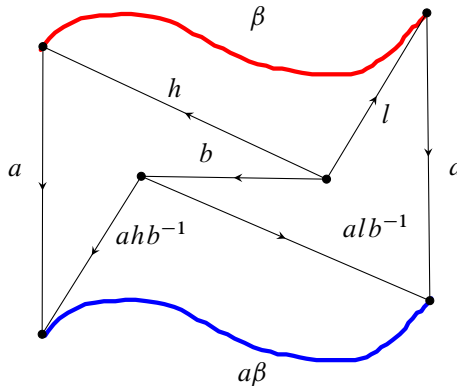


Figure 2

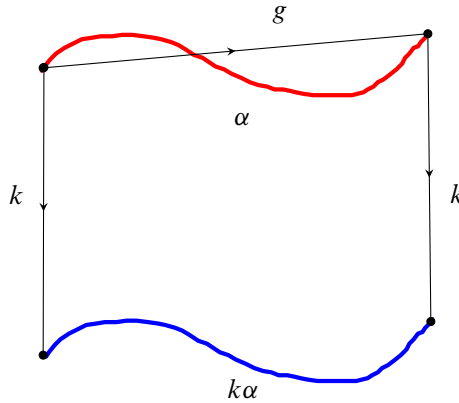


Figure 3

Proof We define an equivariant map $\phi \times \epsilon: (G \times G) \times L_0 \rightarrow G \times L$ by $\phi((a, b)) = a$ and $\epsilon((\beta, h, l)) = (\beta, l^{-1}h)$. This map is an essential equivalence since the map

$$s\pi_1: G \times L_0 \rightarrow L$$

given by $s\pi_1(k, (\beta, h, l)) = (k^{-1}\beta, k^{-1}l^{-1}hk)$ is an open surjection and $G \times G \times L_0$ is given by the pullback of the maps

$$\begin{array}{ccc}
 & G \times L & \\
 & \downarrow (s,t) & \\
 L_0 \times L_0 & \xrightarrow{\epsilon \times \epsilon} & L \times L
 \end{array}
 \quad \square$$

Remark 4.16 We can use our description for the free loop groupoid in the special case of the point groupoid. We obtain that $L(G \times \bullet) = (G \times G) \times (G \times G)$ with the action $(a, b) \cdot (h, l) = (bha^{-1}, bla^{-1})$. This groupoid is equivalent to G acting on itself by conjugation by using the second characterization of the loop groupoid as $G \times L$ with $L = \{(\beta, g) \in X^I \times G \mid \beta(0) = g\beta(1)\}$. In this way, we recover a result of Lupercio and Uribe in [7]. Observe that $L(G \times \bullet) = G \times G$, whereas $P(G \times \bullet) = G \times \bullet$.

5 Based path and loop groupoids

Now that we have defined the free path groupoid of a translation groupoid and have given several equivalent models, we can give an explicit characterization of the various groupoids resulting from fixing certain points. These based groupoids of paths will be of great significance to the groupoid Lusternik–Schnirelmann theory defined in [3] and further studied in [2].

5.1 The groupoid $\Omega_{x,y}$ of paths from x to y

The groupoid of paths from x to y , $\Omega_{x,y}$, is defined as a pullback of the evaluation map $ev: P(G \times X) \rightarrow (G \times G) \times (X \times X)$ and the constant map $x \times y: \mathbf{1} \rightarrow (G \times G) \times (X \times X)$, where $\mathbf{1}$ is the trivial groupoid with one object and one arrow, ie $\mathbf{1} = e \times \bullet$, and $(x \times y)(\bullet) = (x, y)$. That is,

$$\begin{array}{ccc} \Omega_{x,y} & \longrightarrow & P(G \times X) \\ \downarrow & & \downarrow ev \\ \mathbf{1} & \xrightarrow{x \times y} & (G \times G) \times (X \times X) \end{array}$$

Note that by the definition of groupoid pullback, we have that if we take the model of the path groupoid of generalized paths, $P = \text{colim } \phi \times \text{colim } \psi$, then the object space of the pullback is

$$\{((\alpha_1, \dots, \alpha_n, k_1, \dots, k_{n-1}), h, l) \in \text{colim } \psi \times (G \times G) \mid \alpha_1(0) = hx \text{ and } \alpha_n(1) = ly\},$$

ie the objects of $\Omega_{x,y}$ are sequences of paths and arrows $(h, \alpha_1, k_1, \dots, k_n, \alpha_n, l)$ where $s(k_i) = \alpha_{i+1}(r_i)$ for $i = 1, \dots, n - 1$, $t(k_i) = \alpha_i(r_i)$ for $i = 0, \dots, n$, $s(h) = x$ and $s(l) = y$; which are precisely the Haefliger G -paths [6] when restricted to the closed intervals in the subdivision. Note that the sequences in Haefliger paths start and end with *arrows* and not with *paths* like our generalized paths in the free path groupoid defined in Section 4. We recover the original sequence in the Haefliger G -paths when we fix the endpoints x and y in our free generalized paths.

For an equivalent characterization of the groupoid of paths from x to y , we can consider our simplest model for the path groupoid $P'' = G \times X^I$. In this case, we describe the space of objects as $(\Omega_{x,y})_0 = \{(\beta, h, l) \in X^I \times (G \times G) \mid \beta(0) = hx \text{ and } \beta(1) = ly\}$.

These are paths that start at any point in the orbit of x and end at any point in the orbit of y . The space of arrows is the Cartesian product $G \times (\Omega_{x,y})_0$ where the action is given by $g(\beta, h, l) = (g\beta, gh, gl)$; see Figure 4.

Since $(\beta, h, l) \sim (g\beta, gh, gl)$ for all $g \in G$, we can consider $g = h^{-1}$ and we have that all classes $[(\beta, h, l)]$ have a representative of the form (α, k) with $\alpha = h^{-1}\beta$ and $k = h^{-1}l$. Then we can consider the space of objects

$$P_{x,y} = \{(\alpha, k) \in X^I \times G \mid \alpha(0) = x \text{ and } \alpha(1) = ky\}.$$

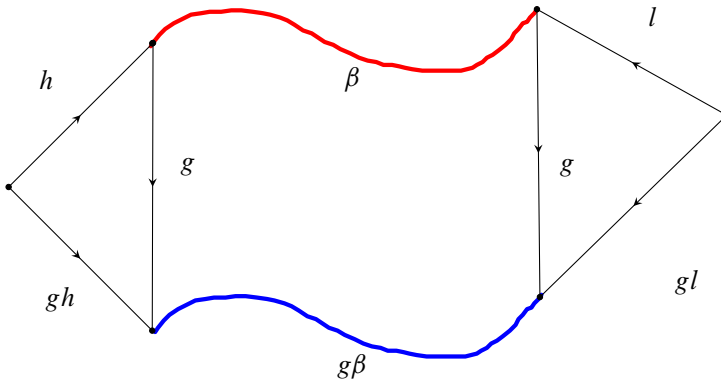


Figure 4

Observe that

$$(\alpha, k) \sim (h^{-1}\beta, e, h^{-1}l) \sim (gh^{-1}\beta, ge, gh^{-1}l) = (g\alpha, g, gk) \sim (e\alpha, e, k) \sim (\alpha, k),$$

so the action is trivial on the space of objects $P_{x,y}$.

Therefore the *groupoid of paths between x and y* is the translation groupoid $\Omega_{x,y} = G \times (\Omega_{x,y})_0$ which is equivalent to the topological space $P_{x,y}$.

5.2 The groupoid Ω_x of based loops

Similarly, we define the based loop groupoid as the groupoid pullback,

$$\begin{array}{ccc} \Omega_x & \longrightarrow & P(G \times X) \\ \downarrow & & \downarrow \text{ev} \\ \mathbf{1} & \xrightarrow{x \times x} & G \times G \times (X \times X) \end{array}$$

where $x \times x$ is the constant map with $(x \times x)(\bullet) = (x, x)$.

That is, the *based loop groupoid* is the translation groupoid $\Omega_x = G \times (\Omega_x)_0$ where the object space is

$$(\Omega_x)_0 = \{(\beta, h, l) \in X^I \times (G \times G) \mid \beta(0) = hx \text{ and } \beta(1) = lx\},$$

ie the space of paths that begin and end at (possibly different) points in the orbit of x . The action is given by $g(\beta, h, l) = (g\beta, gh, gl)$; see Figure 5.

Again, the groupoid Ω_x is equivalent to the topological space

$$P_{x,x} = \{(\alpha, k) \mid \alpha(0) = x \text{ and } \alpha(1) = kx\}.$$

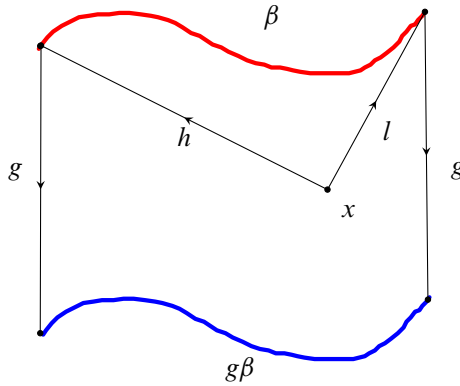


Figure 5

Alternatively, the based loop groupoid Ω_x can be obtained as the groupoid pullback

$$\begin{array}{ccc} \Omega_x & \longrightarrow & L(G \times X) \\ \downarrow & & \downarrow \text{ev}_0 \\ \mathbf{1} & \xrightarrow{x} & G \times X \end{array}$$

where $L(G \times X)$ is the free loop groupoid.

5.3 The groupoid P_x of paths from x

We define the x -based path groupoid as the groupoid pullback

$$\begin{array}{ccc} P_x & \longrightarrow & P(G \times X) \\ \downarrow & & \downarrow \text{ev} \\ \mathbf{1} \times (G \times X) & \xrightarrow{(x, \text{id})} & (G \times G) \times (X \times X) \end{array}$$

where $(x, \text{id}): \mathbf{1} \times (G \times X) \rightarrow (G \times G) \times (X \times X)$ is given by $(x, \text{id})(\bullet, z) = (x, z)$. Then the object space of the pullback P_x is

$$\begin{aligned} (P_x)_0 &= \{(\beta, (h, l), (\bullet, z)) \in X^I \times G \times G \times \mathbf{1} \times X \mid \beta(0) = hx \text{ and } \beta(1) = lz\} \\ &= \{(\beta, (h, l), z) \mid \beta(0) = hx \text{ and } \beta(1) = lz\}. \end{aligned}$$

The group $G \times G$ acts on $(P_x)_0$ by $(g, k)(\beta, (h, l), z) = (g\beta, (gh, glk^{-1}), kz)$; see Figure 6.

The x -based path groupoid is the translation groupoid $P_x = (G \times G) \times (P_x)_0$.

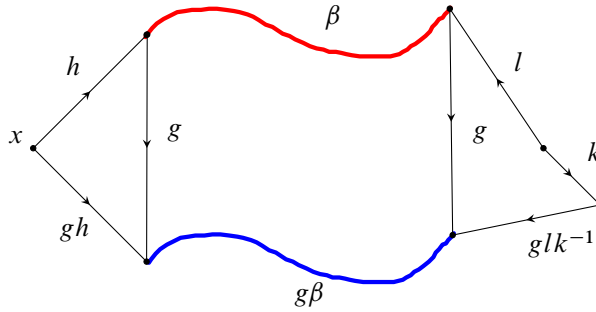


Figure 6

Observing that the equivalence class of each $(\beta, (h, l), z) \in (P_x)_0$ contains an element of the form $(\alpha, g, w) \in X^I \times G \times X$ we have that the based path groupoid P_x is equivalent to $G \ltimes P$ where $P = \{(\alpha, g, w) \mid \alpha(0) = x \text{ and } \alpha(1) = gw\}$ and the action is given by $k(\alpha, g, w) = (\alpha, gk^{-1}, kw)$. Figure 7 depicts an arrow $(k, (\alpha, g, w)) \in G \times P$ between (α, g, w) and (α, gk^{-1}, kw) .

The x -based path groupoid P_x is not in general equivalent to a topological space.

Given points $x, y \in X$, our various path groupoids are related by

$$\begin{array}{ccccccc}
 & & \Omega_{x,y} & \xrightarrow{\quad} & \mathbf{1}_y & & \\
 & & \downarrow & & \downarrow & & \\
 \Omega_x & \longrightarrow & P_x & \longrightarrow & G \ltimes X^I & \xrightarrow{\text{ev}_1} & G \ltimes X \\
 & & \downarrow & & \downarrow \text{ev}_0 & & \\
 & & \mathbf{1}_x & \longrightarrow & G \ltimes X & &
 \end{array}$$

where $\mathbf{1}_x = e \ltimes x$ and $\mathbf{1}_y = e \ltimes y$ and all diagrams are commutative up to a natural transformation.

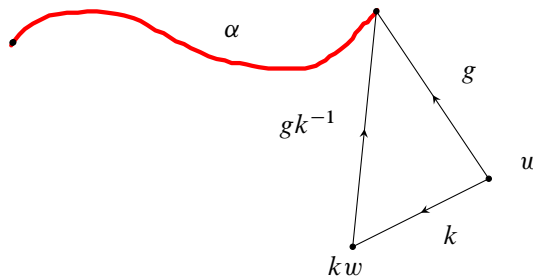


Figure 7

5.4 Examples

We will illustrate in this section the concepts described in the previous sections by calculating various path groupoids in some particular cases.

5.4.1 Topological spaces The free path groupoid of the topological space X is $P(e \times X) = e \times X^I = X^I$ and the free loop groupoid is $L(e \times X) = e \times L$ where $L = \{\alpha \in X^I \mid \alpha(0) = \alpha(1)\}$. In this way we recover the classical free path and loop spaces of a topological space. Likewise, the based path and loop groupoids also coincide with the classical ones for topological spaces.

5.4.2 Groups For a point groupoid $G \times \bullet$ we have shown before that the path groupoid is itself and the loop groupoid is $(G \times G) \times (G \times G)$ with the action $(a, b) \cdot (h, l) = (bha^{-1}, bla^{-1})$, which is equivalent to G acting on itself by conjugation; that is, $L(G \times \bullet) = G \times G$ and $P(G \times \bullet) = G \times \bullet$. The based loop groupoid is the unit groupoid G , as a discrete topological space. The based path groupoid of paths emanating from \bullet is $G \times G$.

5.4.3 Free actions If G acts freely on a topological space X , we observe that the groupoid $G \times X$ and the topological space X/G are Morita equivalent. Then, we have that $P(G \times X) = P(e \times X/G) = e \times (X/G)^I = (X/G)^I$ and the free loop groupoid is $L(G \times X) = L(X/G)$ where $L(X/G)$ is the free loop space of the topological space X/G . In the same way, we have that the based groupoids coincide with the ones of the topological space X/G .

5.4.4 Orbifolds We proved that for developable orbifolds $G \times X$, the free path groupoid is $P(G \times X) = G \times X^I$ and the free loop groupoid is $L(G \times X) = G \times L$ where $L = \{(\alpha, g) \in X^I \times G \mid \alpha(0) = g\alpha(1)\}$. Also, the groupoid of paths between x and y is the topological space $P_{x,y} = \{(\alpha, k) \in X^I \times G \mid \alpha(0) = x \text{ and } \alpha(1) = ky\}$, the groupoid of based loops is the topological space $P_{x,x} = \{(\alpha, k) \mid \alpha(0) = x \text{ and } \alpha(1) = kx\}$ and the groupoid of based paths from x is the translation groupoid $P_x = (G \times G) \times (P_x)_0$.

6 Homotopy

We will define in this section a notion of homotopy based on the explicit description of the path groupoid $P(G \times X)$ given in the previous section. This will provide a concrete alternative to the more abstract presentation given by Noohi in [10; 11] for stacks.

6.1 Natural transformations for translation groupoids

The equivariant maps $\varphi \times f: K \times Z \rightarrow G \times X$ and $\psi \times g: K \times Z \rightarrow G \times X$ are equivalent by a natural transformation if there exists a K -map $\gamma: Z \rightarrow G$ such that $\gamma(z)f(z) = g(z)$ for all $z \in Z$ where both Z and G are K -spaces considering the action of K on G ,

$$K \times G \rightarrow G, \quad (k, g) \mapsto \psi(k)g\varphi(k)^{-1}.$$

Therefore $\varphi \times f \sim \psi \times g$ if there exists $\gamma: Z \rightarrow G$ such that

- (1) $\gamma(z)f(z) = g(z)$ for all $z \in Z$, and
- (2) $\gamma(kz) = \psi(k)\gamma(z)\varphi(k)^{-1}$ for all $k \in K$.

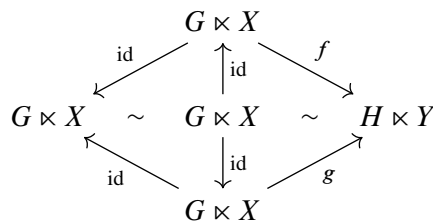
If Z is connected, then γ is a constant map since G is discrete. Then $\varphi \times f \sim \psi \times g$ if there exists $h \in G$ such that $hf(z) = g(z)$ for all $z \in Z$ and $h = \psi(k)h\varphi(k)^{-1}$ for all $k \in K$. Then $g = hf$ and $\psi = h\varphi h^{-1}$. In other words, $\psi(k)$ is conjugated to $\varphi(k)$ for all $k \in K$.

In addition, if G is abelian, then $\varphi \times f \sim \psi \times g$ if $g = hf$ for some $h \in G$ and $\varphi = \psi$. If $X = Z = \bullet$, then $\varphi \times \bullet \sim \psi \times \bullet$ if and only if φ and ψ are conjugate, $\varphi = h^{-1}\psi h$. In particular, when the group acting is abelian we have that two maps between point groupoids are equivalent by a natural transformation only if they are equal.

We give now a characterization of 2-isomorphism for strict maps. Namely, if two strict maps are 2-isomorphic then when composed with an essential equivalence they are equivalent by a natural transformation, and if two strict maps are equivalent by a natural transformation then they are 2-isomorphic as generalized maps.

Proposition 6.1 *If f and g are equivalent by a natural transformation, then $f \Rightarrow g$ as generalized equivariant maps.*

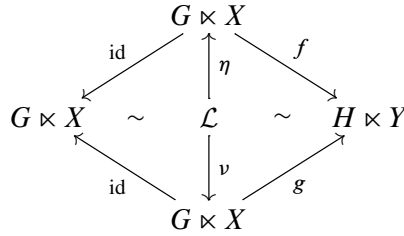
Proof Just consider the essential equivalences η and ν as identity maps and the following diagram is commutative up to natural transformations since $f \sim g$:



□

Proposition 6.2 *If two strict maps $f: G \times X \rightarrow H \times Y$ and $g: G \times X \rightarrow H \times Y$ are 2-isomorphic, then there exists an essential equivalence $\eta: \mathcal{L} \rightarrow G \times X$ such that $f\eta \sim g\eta$.*

Proof We have that there exist essential equivalences η, ν such that the diagram



commutes up to natural transformation. That is, $\eta \sim \nu$ and $f\eta \sim g\nu$. Therefore, $f\eta \sim g\eta$. □

Proposition 6.3 *If $(\epsilon, f) \Rightarrow (\sigma, g)$, then there exist essential equivalences ν and η such that $f\nu \Rightarrow g\eta$.*

Proof By definition of 2-isomorphism, there are essential equivalences ν and η such that $f\nu \sim g\eta$. The result follows from Proposition 6.1. □

Proposition 6.4 *If $f \Rightarrow g$, then $(\epsilon, f) \Rightarrow (\sigma, g)$ for all essential equivalences ϵ and σ with $\epsilon \sim \sigma$.*

6.2 Diagonal map

We will consider the pullback of the unique morphism $G \times X \xrightarrow{c} \mathbf{1}$ with itself, where $\mathbf{1}$ is the terminal object in \mathbf{MTopG} . This pullback defines the product and then by the universal property we obtain the definition of the diagonal map. Then, the path groupoid will be a factorization of that diagonal.

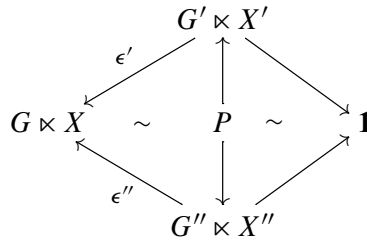
Definition 6.5 [5] An object T in a bicategory \mathbf{B} is terminal if the category $\mathbf{B}[C, T]$ is equivalent to the terminal category for every object C in \mathbf{B} . A terminal object is unique up to equivalence when it exists.

The trivial groupoid $\mathbf{1} = e \times \bullet$ is the terminal object in the bicategory of translation groupoids \mathbf{MTrG} since the category $\mathbf{MTrG}[G \times X, \mathbf{1}]$ is equivalent to the category $\mathbf{1}$.

Indeed, the objects in the category $\text{MTrG}[G \times X, \mathbf{1}]$ are generalized maps and the arrows are classes of diagrams. We can see that all objects are related by an arrow, ie $\text{MTrG}[G \times X, \mathbf{1}]$ is the pair groupoid. Given two generalized maps,

$$G \times X \xleftarrow{\epsilon'} G' \times X' \xrightarrow{c'} \mathbf{1} \quad \text{and} \quad G \times X \xleftarrow{\epsilon''} G'' \times X'' \xrightarrow{c''} \mathbf{1},$$

we can see that they are equivalent, ie



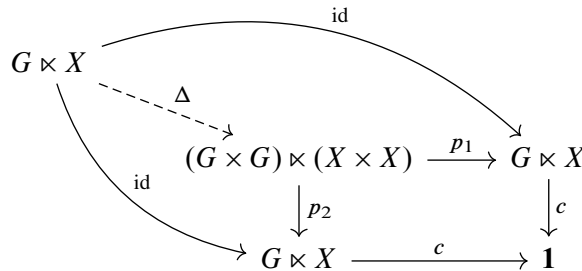
by considering P as the pullback of ϵ' and ϵ'' . In particular, the strict constant map $G \times X \xrightarrow{c} \mathbf{1}$ is the (unique up to 2-isomorphism) map to the terminal object.

Let us now consider the pullback of this constant map with itself which defines the product

$$\begin{array}{ccc} G \times G \times (X \times X) & \longrightarrow & G \times X \\ \downarrow & & \downarrow c \\ G \times X & \xrightarrow{c} & \mathbf{1} \end{array}$$

The product $(G \times G) \times (X \times X)$ of the object $G \times X$ with itself is unique up to equivalence.

By the universal property of the pullback, there exists a map Δ that makes the two triangles commutative up to natural transformation



The map $\Delta: G \times X \rightarrow (G \times G) \times (X \times X)$ is the *diagonal map*. Its explicit definition on objects is $\Delta(x) = (x, x)$ and on arrows, $\Delta(g, x) = (g, g, x, x)$. The diagonal map is defined up to 2-isomorphism.

Remark 6.6 The diagonal defined in [1] is 2-isomorphic to this one.

Definition 6.7 The *evaluation map* $ev: G \times X^I \rightarrow (G \times G) \times (X \times X)$ is given by $ev(g, \alpha) = (g, g, \alpha(0), \alpha(1))$.

We have that the diagonal map factors through the path groupoid as expected.

Proposition 6.8 *There is a factorization of the diagonal map Δ*

$$\begin{array}{ccc}
 G \times X & \xrightarrow{\Delta} & (G \times G) \times (X \times X) \\
 & \searrow k & \nearrow e \\
 & & G \times X^I
 \end{array}$$

where k and e are generalized maps.

Proof Let k be the functor $G \times X \rightarrow G \times X^I$ given by $x \rightsquigarrow \alpha_x$ on objects, and $(g, x) \rightsquigarrow (g, \alpha_x)$, where $\alpha_x: I \rightarrow X$ is a constant path at $x \in X$, and let e be the evaluation map, $e = ev$. Then, we have that the composition $e \circ k$ is equivalent by a natural transformation to the diagonal Δ . □

6.3 Homotopic maps

We will now give an explicit characterization of the homotopy between generalized maps.

Definition 6.9 Two generalized maps,

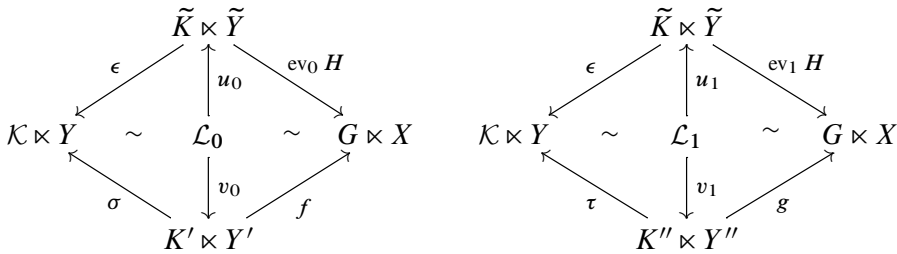
$$K \times Y \xleftarrow{\sigma} K' \times Y' \xrightarrow{f} G \times X \quad \text{and} \quad K \times Y \xleftarrow{\tau} K'' \times Y'' \xrightarrow{g} G \times X,$$

are *homotopic* if there is a generalized map $K \times Y \xleftarrow{\epsilon} \tilde{K} \times \tilde{Y} \xrightarrow{H} G \times X$ such that the following diagram commutes up to 2-isomorphism:

$$\begin{array}{ccccc}
 G \times X & \xleftarrow{ev_0} & G \times X^I & \xrightarrow{ev_1} & G \times X \\
 & \searrow f & \uparrow H & & \nearrow g \\
 & & \tilde{K} \times \tilde{Y} & & \\
 & & \downarrow \epsilon & & \\
 K' \times Y' & & & & K'' \times Y'' \\
 & \searrow \sigma & & \swarrow \tau & \\
 & & K \times Y & &
 \end{array}$$

This means that the generalized map (σ, f) is isomorphic to the generalized map $(\epsilon, ev_0 \circ H)$ and (τ, g) is isomorphic to $(\epsilon, ev_1 \circ H)$.

That is, (σ, f) is homotopic to (τ, g) if there exists (ϵ, H) and two commutative diagrams up to natural transformations,



where \mathcal{L}_i is a translation groupoid, and u_i and v_i are equivariant essential equivalences for $i = 0, 1$. We will denote this homotopy between equivariant generalized maps by \simeq .

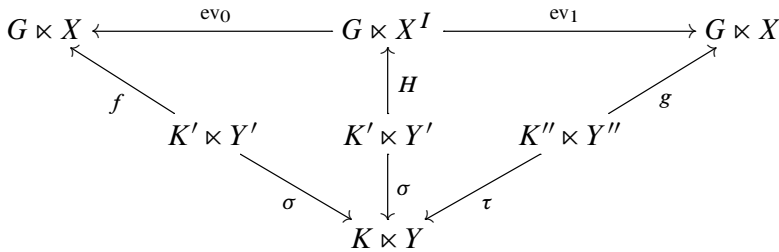
Remark 6.10 $(\sigma, f) \simeq (\tau, g)$ if there exists (ϵ, H) and essential equivalences u_0, u_1, v_0 and v_1 such that

$$f v_0 \sim \text{ev}_0 H u_0 \quad \text{and} \quad g v_1 \sim \text{ev}_1 H u_1$$

with $\epsilon u_0 \sim \sigma v_0$ and $\epsilon u_1 \sim \tau v_1$.

Proposition 6.11 If $(\sigma, f) \Rightarrow (\tau, g)$, then $(\sigma, f) \simeq (\tau, g)$.

Proof Consider $H = i_X \circ f$ where i_X is the inclusion of X in X^I given by $i_X(x) = \alpha_x$ with α_x being the constant map $\alpha_x(t) = x$ for all $t \in I$. Then the following diagram is commutative up to 2-isomorphism:



The first triangle is an equality and the second is commutative since $(\sigma, f) \Rightarrow (\tau, g)$. \square

Remark 6.12 Let f and g be strict maps. Following the characterization for isomorphism of strict maps given in Proposition 6.2 and the definition of groupoid homotopy, we have that $f \simeq g$ if there exists a generalized map (ϵ, H) and essential equivalences η and ν such that $f \epsilon \eta \sim \text{ev}_0 H \eta$ and $g \epsilon \nu \sim \text{ev}_1 H \nu$.

Proposition 6.13 *Let f and g be strict maps.*

- (1) *If f and g are ψ -equivariantly homotopic maps, then $f \simeq g$ as generalized equivariant maps.*
- (2) *If f and g are equivalent by a natural transformation, then $f \simeq g$ as generalized equivariant maps.*

Proof (1) Let $H : Y \rightarrow X^I$ be the ψ -equivariant homotopy, ie $H_t(ky) = \psi(k)H_t(y)$. Then the following diagram is commutative:

$$\begin{array}{ccccc}
 G \times X & \xleftarrow{\text{ev}_0} & G \times X^I & \xrightarrow{\text{ev}_1} & G \times X \\
 & \searrow f & \uparrow H & \nearrow g & \\
 & & K \times Y & &
 \end{array}$$

- (2) This follows from Propositions 6.1 and 6.11. □

Therefore our definition of homotopy generalizes both the notion of natural transformation and the notion of equivariant homotopy.

Proposition 6.14 *If $(\epsilon, f) \simeq (\sigma, g)$ then there exist essential equivalences a and b such that $fa \simeq gb$ as strict maps.*

Proof Since we have a homotopy between generalized maps, we know that there exists (δ, H) and essential equivalences u_0, v_0, u_1 and v_1 such that

$$fv_0 \sim \text{ev}_0 H u_0, \quad gv_1 \sim \text{ev}_1 H u_1, \quad \delta u_0 \sim \epsilon v_0, \quad \delta u_1 \sim \sigma v_1.$$

Take $a = v_0(u_0)^{-1}\delta^{-1}$ and $b = v_1(u_1)^{-1}\delta^{-1}$. Then fa and gb are homotopic. □

Proposition 6.15 *The path groupoid $G \times X^I$ is homotopy equivalent to the groupoid $G \times X$. The evaluation $e_1 : G \times X^I \rightarrow G \times X$ is a homotopy equivalence.*

Proof Consider the map $H : G \times X^I \rightarrow G \times (X^I)^I$ such that $H(\alpha) = \lambda$ with

$$\lambda : I \rightarrow X^I, \quad \lambda(t) = \alpha(r + t - rt).$$

We have the commutative diagram

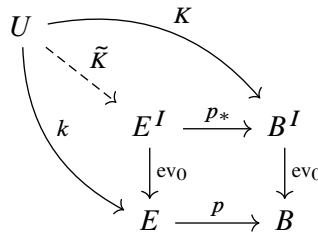
$$\begin{array}{ccccc}
 G \times X^I & \xleftarrow{\text{ev}_0} & G \times (X^I)^I & \xrightarrow{\text{ev}_1} & G \times X^I \\
 & \searrow \text{id} & \uparrow H & \nearrow i \circ e_1 & \\
 & & G \times X^I & &
 \end{array}$$

showing that $i \circ e_1$ is homotopic to the identity map. □

7 Fibrations

We recall the definition of fibration for topological spaces given as a dualization of the notion of cofibration.

Definition 7.1 [8; 18] A map $p: E \rightarrow B$ is a fibration if for all spaces U with $ev_0 \circ K = p \circ k$ in the diagram

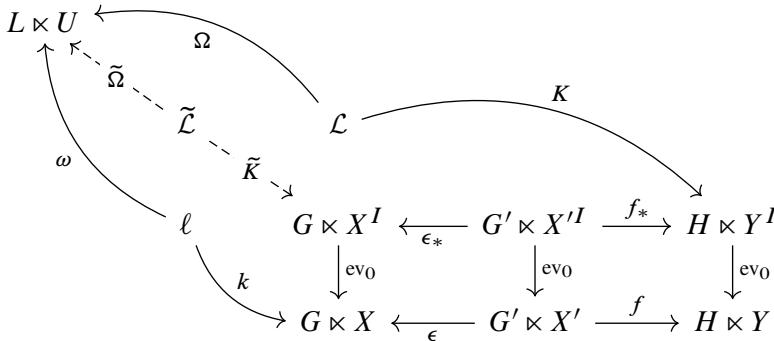


there exists \tilde{K} that makes the diagram commute.

We want to introduce a notion of fibration for generalized maps. First, let us note that a strict equivariant map $\varphi \times f: G \times X \rightarrow H \times Y$ induces a map $\varphi_* \times f_*: G \times X^I \rightarrow H \times Y^I$ by $f_*(\alpha) = f \circ \alpha$ for all $\alpha \in X^I$ and $\varphi_* = \varphi$. We proved in Proposition 4.12 that if $\epsilon: G \times X \rightarrow H \times Y$ is an essential equivalence, then $\epsilon_*: G \times X^I \rightarrow H \times Y^I$ is an essential equivalence as well.

Then every generalized map $G \times X \xleftarrow{\epsilon} G' \times X' \xrightarrow{f} H \times Y$ induces a generalized map $G \times X^I \xleftarrow{\epsilon_*} G' \times X'^I \xrightarrow{f_*} H \times Y^I$ between the path groupoids.

Definition 7.2 A generalized map $G \times X \xleftarrow{\epsilon} G' \times X' \xrightarrow{f} H \times Y$ is a *groupoid fibration* if for all translation groupoids $L \times U$ with $ev_0 \circ (\Omega, K) \Rightarrow (\omega, k) \circ (\epsilon, f)$ in the diagram



there exists $(\tilde{\Omega}, \tilde{K})$ that makes the diagram commute up to 2-isomorphism.

Since a 2–isomorphism between strict maps induces a 2–isomorphism between the induced maps between their path groupoids, being a fibration is a property invariant under 2–isomorphism.

Proposition 7.3 Consider 2–isomorphic maps

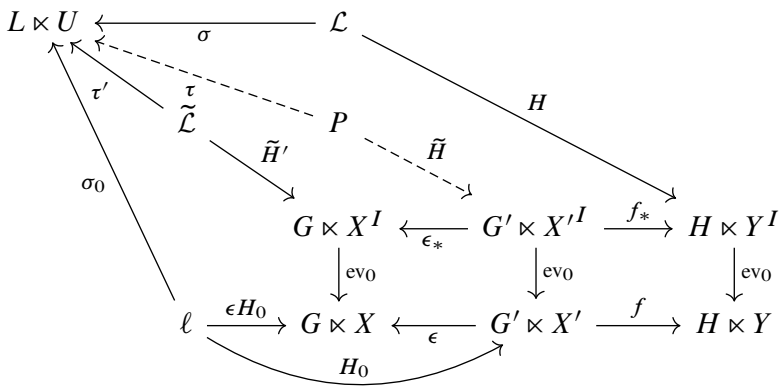
$$f : G \times X \rightarrow H \times Y \quad \text{and} \quad g : G \times X \rightarrow H \times Y,$$

$f \Rightarrow g$. Then f is a fibration if and only if g is a fibration.

We will see that for (ϵ, f) to be a groupoid fibration it is necessary and sufficient that the right leg of the span is a groupoid fibration (considered as a generalized map with identity as a left leg).

Proposition 7.4 A generalized map $G \times X \xleftarrow{\epsilon} G' \times X' \xrightarrow{f} H \times Y$ is a groupoid fibration if and only if $f : G' \times X' \rightarrow H \times Y$ is a groupoid fibration.

Proof If the generalized map (ϵ, f) is a groupoid fibration, then there exists (τ', \tilde{H}') that makes the diagram



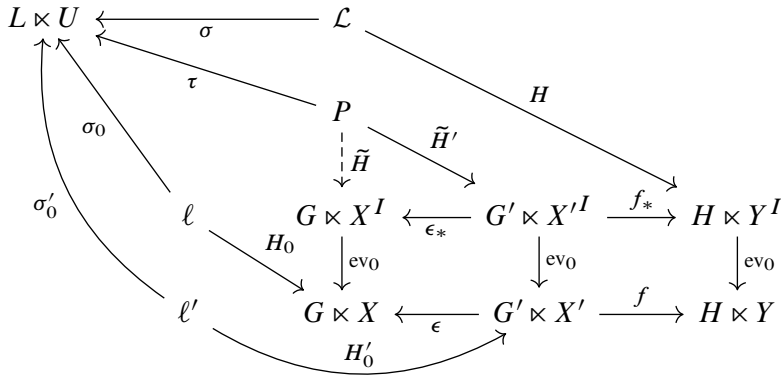
commute up to 2–isomorphism.

Let P be the pullback

$$\begin{array}{ccc} P & \xrightarrow{\tilde{H}''} & G' \times X'^I \\ \downarrow \epsilon'_* & & \downarrow \epsilon_* \\ \tilde{\mathcal{L}} & \xrightarrow{\tilde{H}'} & G \times X^I \end{array}$$

Take $\tau = \tau' \epsilon'_*$ and $\tilde{H} = \tilde{H}''$. Then $f : G' \times X' \rightarrow H \times Y$ is a groupoid fibration.

Conversely, if f is a groupoid fibration then we have this commutative diagram

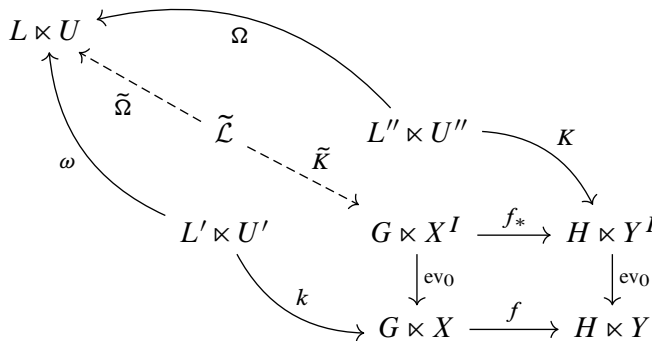


where $(\sigma'_0, H'_0) = \epsilon \circ (\sigma_0, H_0)$. Now take $\tilde{H} = \epsilon_* \tilde{H}'$ and $\tau = \tau'$. Therefore, (ϵ, f) is a fibration. □

Then, the test to decide if a generalized map is a groupoid fibration amounts to check the definition of groupoid fibration with a strict map. Moreover, we know that any generalized map $L \times U \leftarrow \ell \rightarrow G \times X$ is equivalent to a generalized map of the form $L \times U \leftarrow L' \times U' \rightarrow G \times X$, where L' may be chosen as $L \times G$ and the group homomorphisms are the appropriate projections onto L and G [14].

The groupoid fibration definition specializes to the following:

Definition 7.5 A strict map $f : G \times X \rightarrow H \times Y$ is a groupoid fibration if for all translation groupoids $L \times U$ with $\text{ev}_0 \circ (\Omega, K) \Rightarrow f \circ (\omega, k)$ in the diagram



there exists $(\tilde{\Omega}, \tilde{K})$ that makes the diagram commute up to 2-isomorphism.

In other words, f is a groupoid fibration if for all commutative diagrams

$$\begin{array}{ccccc}
 & & (L'') \times U'' & & \\
 & \swarrow \Omega & \uparrow \eta & \searrow \text{ev}_0 \circ K & \\
 L \times U & & & & H \times Y \\
 & \swarrow \omega & \downarrow \nu & \searrow f \circ k & \\
 & & (L \times G) \times U' & &
 \end{array}$$

there exists $(\tilde{\Omega}, \tilde{K})$ such that the following diagrams commute:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 L'' & \xleftarrow{\tilde{\Omega}} & \tilde{\mathcal{L}} \\
 & \searrow \omega & \downarrow \nu' \\
 & & (L \times G) \times U'
 \end{array}
 &
 &
 \begin{array}{ccc}
 \tilde{\mathcal{L}} & \xrightarrow{\text{ev}_0 \circ \tilde{K}} & G \times X \\
 \uparrow \eta' & \swarrow \sim & \nearrow k \\
 & &
 \end{array}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{ccc}
 L \times U & \xleftarrow{\tilde{\Omega}} & \tilde{\mathcal{L}} \\
 & \searrow \Omega & \downarrow \nu'' \\
 & & (L \times H) \times U''
 \end{array}
 &
 &
 \begin{array}{ccc}
 \tilde{\mathcal{L}} & \xrightarrow{f_* \circ \tilde{K}} & H \times Y^I \\
 \uparrow \eta'' & \swarrow \sim & \nearrow K \\
 & &
 \end{array}
 \end{array}$$

Proposition 7.6 *The evaluation map $\text{ev}_0: G \times X^I \rightarrow G \times X$ is a groupoid fibration.*

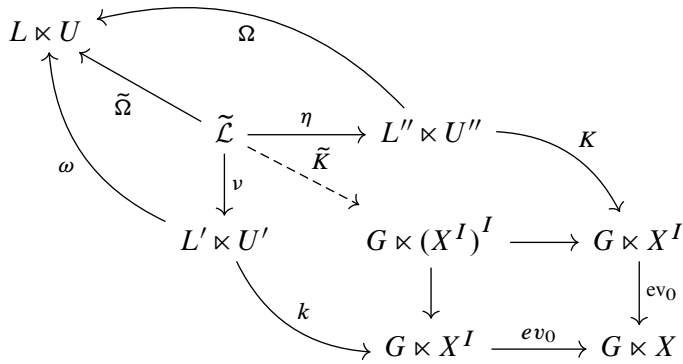
Proof For all translation groupoids $L \times U$ making the following diagram commutative up to 2-isomorphism, we will construct the required generalized map $(\tilde{\Omega}, \tilde{K})$:

$$\begin{array}{ccccc}
 L \times U & & & & L'' \times U'' \\
 & \swarrow \Omega & & & \searrow K \\
 & & \tilde{\mathcal{L}} & & \\
 & \swarrow \tilde{\Omega} & \downarrow \tilde{K} & & \\
 & & L' \times U' & & G \times (X^I)^I \\
 & \swarrow \omega & \downarrow k & \longrightarrow & G \times X^I \\
 & & & \searrow \text{ev}_0 & \downarrow \text{ev}_0 \\
 & & & & G \times X
 \end{array}$$

Since there is a 2-isomorphism between the generalized maps $(\Omega, \text{ev}_0 K)$ and $(\omega, \text{ev}_0 k)$, we know that there exists a groupoid $\tilde{\mathcal{L}}$ and essential equivalences ν and η such that the following diagram commutes up to natural transformations:

$$\begin{array}{ccccc}
 & & L'' \times U'' & & \\
 & \swarrow \Omega & \uparrow \eta & \searrow \text{ev}_0 \circ K & \\
 L \times U & & \tilde{\mathcal{L}} & & G \times X \\
 & \swarrow \omega & \downarrow \nu & \searrow \text{ev}_0 \circ k & \\
 & & L' \times U' & &
 \end{array}$$

We take $\tilde{\Omega} = \Omega\eta$ and will construct a map $\tilde{K}: \tilde{\mathcal{L}} \rightarrow G \times (X^I)^I$ such that the following diagram commutes up to natural transformations:



Consider the groupoid pullback

$$\begin{array}{ccc}
 P & \xrightarrow{\pi_1} & G \times X^I \\
 \downarrow \pi_2 & & \downarrow \text{ev}_0 \\
 G \times X^I & \xrightarrow{\text{ev}_0} & G \times X
 \end{array}$$

where P is the translation groupoid

$$P = (G \times G) \times (X^I \times_X X^I \times_X G)$$

with $X^I \times_X X^I \times_X G = \{(\alpha_1, \alpha_2, k) \mid k\alpha_1(0) = \alpha_2(0)\}$. We observe that in fact P is equivalent to $\text{Map}(I_{S_2}, G \times X)$. To show this equivalence, we construct first a functor $\bar{K}: P \rightarrow G \times (X^I \times_X X^I)$, where $X^I \times_X X^I = X^{I \vee I}$ is the pullback of the diagram

$$\begin{array}{ccc}
 & X^I & \\
 & \downarrow \text{ev}_0 & \\
 X^I & \xrightarrow{\text{ev}_0} & X
 \end{array}$$

given by $\bar{K}(\alpha_1, \alpha_2, k) = (k\alpha_1, \alpha_2)$ on objects and $\bar{K}(g_1, g_2) = g_2$ on morphisms.

Since $(g_1, g_2) \cdot (\alpha_1, \alpha_2, k) = (g_1\alpha_1, g_2\alpha_2, g_2k g_1^{-1})$ and

$$\bar{K}(g_1\alpha_1, g_2\alpha_2, g_2k g_1^{-1}) = (g_2k g_1^{-1}g_1\alpha_1, g_2\alpha_2) = (g_2k\alpha_1, g_2\alpha_2) = g_2(k\alpha_1, \alpha_2),$$

we can see that this is just a special case of the equivalences of the path groupoid models from Section 4,

$$\text{Map}(I_{S_2}, G \times X) \cong P \sim G \times X^{I \vee I} \cong G \times X^I.$$

We observe that the diagram of functors

$$\begin{array}{ccccc}
 G \times X^I & \xleftarrow{j_1^*} & G \times X^{I \vee I} & \xrightarrow{j_2^*} & G \times X^I \\
 & \searrow \pi_1 & \uparrow \bar{K} & \swarrow \pi_2 & \\
 & & P & &
 \end{array}$$

commutes up to natural transformations since the right-hand side commutes on the nose and the left-hand side commutes up to a natural transformation. Here $j_1: I \rightarrow I \vee I$ and $j_2: I \rightarrow I \vee I$ are the natural maps for the coproduct of pointed spaces

$$\begin{array}{ccccc}
 I & \xrightarrow{i_1} & I \times I & \xleftarrow{i_2} & I \\
 & \searrow j_1 & \downarrow \pi & \swarrow j_2 & \\
 & & I \vee I & &
 \end{array}$$

where $i_1(t) = (t, 0)$, $i_2(s) = (0, s)$ and $\pi: I \times I \rightarrow I \vee I$ is a deformation retract. Therefore, we have the commutative diagram

(2)

$$\begin{array}{ccccccc}
 P & & & & & & \\
 \downarrow \bar{K} & & \searrow \pi_1 & & \searrow j_1^* & & \\
 G \times X^{I \vee I} & & & & & & \\
 \downarrow \pi^* & & \searrow j_2^* & & \searrow i_1^* & & \\
 G \times X^{I \times I} & & & & G \times X^I & & \\
 \downarrow i_2^* & & \searrow \text{ev}_0 & & \downarrow \text{ev}_0 & & \\
 G \times X^I & \xrightarrow{\text{ev}_0} & G \times X & & & &
 \end{array}$$

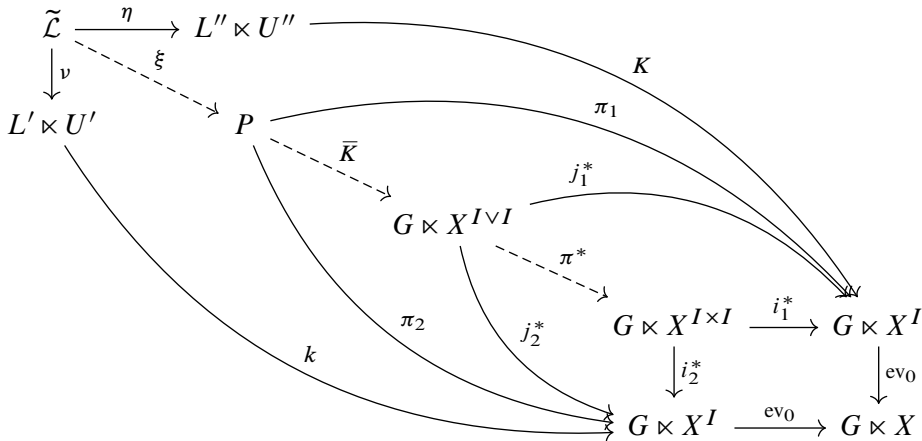
Now, by the universal property of the groupoid pullback, there exists a functor $\xi: \mathcal{L} \rightarrow P$ such that the diagram

(3)

$$\begin{array}{ccc}
 \mathcal{L} & \xrightarrow{K\eta} & P \\
 \downarrow \xi & & \downarrow \\
 G \times X^I & \xrightarrow{\text{ev}_0} & X
 \end{array}$$

commutes up to natural transformation.

Now, we put together diagrams (2) and (3) to obtain



and define $\tilde{K} = \pi^* \circ \bar{K} \circ \xi$. □

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